AN ASPECT OF A DIDACTICAL PATH FOR APPROACHING THE CONCEPT OF FUNCTION: THE QUALITATIVE INTERPRETATION OF A GRAPH

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Abstract: Here we shall look at the results of some on-going research being carried out over the triennium of middle school. It aims to approach the concept of function in the broadest sense through the co-ordination of various levels of representation (verbal, numerical-tabular, graphic and algebraic). In particular, we shall concentrate on the behaviour and difficulties encountered by seventh and eighth grade pupils with regard to some specific activities: they are asked to choose a graph from a selection which corresponds to a given physical phenomenon described verbally with the overall aim of co-ordinating the variational/qualitative and pointwise approaches.

THEORETICAL FRAME

There are many teaching/learning problems surrounding the concept of function. These are bound up in the complexity of its history (for the historic aspects see Youschkevitch 1981), which are reflected in the partial or distorted views of the concept held by students as well as in the difficulties they meet as shown in various studies on the matter (see, for example, classical papers such as Eisemberg 1991; Breidenbach *et al.* 1992; Monk 1992; Vinner 1992 or De Marois & Tall 1999). Other studies examine the constructive contribution made by the use of computers and by the dynamic visualisation of function graphs used to overcome or at least lessen the classic problems of co-ordination between the various registers of representation (Schwartz & Dreyfus 1995; Slavit 1997). Others have concentrated on the possibilities offered by the emerging technologies in order to work on function as a mathematic object (Dagher & Artigue 1993; Kieran 1994, 1998; Borba & Confrey 1996). However, these studies are pitched at a higher scholastic level, usually the last three years of secondary school or above. Studies on problems encountered by pupils approaching this concept at lower scholastic levels are far rarer.
We must remember that the middle school curriculum in our country (pupils between 11 and 14 years old) emphasise the study of relationships and the use of the algebraic language with which to codify them, focusing the attention on functions as an instrument with which to form models for simple physical phenomena. In this approach, a view of functions as a functional correspondence -which can be represented algebrically- between different magnitudes prevails. This view is closer to the traditional concept than the modern one of arbitrary functional correspondence which is left to the high school years. Furthermore, we must consider that, owing to the lack of any consistent tradition of teaching on this topic in early stages, often the teachers themselves lament uncertainties and difficulties when teaching it (Even, 1993).
At the middle school level, it seems didactically important to approach the problem in such a way that might encourage a flexible view of the concept in the pupils, examining various aspects of the matter while laying down the founding stones for later development. In particular, the use of the graph, which has always been at the origin of the concept of function, may in fact constitute an epistemological obstacle to the understanding of the concept in its most modern meaning if not handled wisely. We believe that in order to develop a true knowledge of a subject, it is necessary for the student to distinguish between mathematical concepts and their representations. From a didactical point of view, this sets the problem of how to lead the student to put this distinction into practice not after but *during* the learning process. Faced with the multiplicity of representative levels of a function, the graphic level clearly plays a fundamental role (Mavarech & Kramarsky 1997; Vinner 1992), yet it is only through the co-ordination of different registers (verbal, tabular, algebraic) that one can achieve true conceptualisation (Duval 2000).

HYPOTHESIS OF THE RESRARCH

The research concerns some results of a complex didactic plan which we have designed for the three years of middle school (see table 1) and forms part of a wider innovative project concerning first steps in algebra (Malara & Iaderosa 1999). The aim of the research is to analyse key difficulties that emerge in its teaching/learning process and to produce an innovative prototype which would also be useful in teacher training, in line with the tradition of research of our country (Arzarello & Bartolini Bussi 1998). Initially, this plan (put into practice in classes of the three different years taught by the same teacher, R. Iadarosa) provides an approach to the graph as an instrument representing real-life phenomena which the pupils are already familiar with or, better still, have some direct control over. It includes both tasks of

Table 1

Didactical framework in the triennium of middle school
sixth grade: highlighting the difficulties that the weaker pupils usually have at the beginning of middle school with the following activities: reading and using literal symbols (letter as generalization of number), discovering relationships between numbers, expressing verbally; co-ordinating the numerical/literal register with the verbal register and vice-versa; verbal interpretation and reading data from graphs that represent non-codified rules. Seventh grade: discovering and expliciting relationships in the verbal and in the numerical/algebraic register; recognising and interpreting graphs on the basis of what they say about the variability of the quantities examined, without formalised rules; co-ordinating the verbal and algebraic registers with the graphic register: representing relationships, expressed in many forms (from the implicit to the explicit form, from direct to inverse, etc.); analysing and comparing the graphic representations derived from different formulations of the same algebraic relation; first attempts at formalising relations algebraically through the observation of numerical data; combining rules and graphs in an intuitive, non-formalized way, interpreting graphs that mainly represent physical phenomena. Eighth grade: algebraic transformation of mathematical functions in the three possible forms, the two explicit and the implicit one; recognising some fundamental forms of functions (from a structural point of view): constant ratio, constant sum, constant product, and matching each of these forms with a particular graph; analysis of each of these classes of functions from the algebraic point of view; geometrical interpretation of each element of the formula (numerical coefficient, its sign, etc.).

constructing graphs from tables of data and the relationship between the magnitudes in play, and in particular the analysis of the tendency of the graph aimed at making a qualitative interpretation of the variational link represented. This plan also includes comparisons between different graphs relating to the same phenomenon which highlight functional links between different magnitudes concerned. Our hypothesis is that only through this baggage of experience can the pupils move flexibly and consistently between variational/qualitative and pointwise approaches, as Tierney *et al.* (1992) also underlined and more generally can they grasp the sense and importance of the study of the most elementary algebraic forms on which various phenomena are based.

Let us make it clear that, generally speaking, the working method adopted in the classes is based on the systematic use of written verbalization, on the analysis of strategies put into practice by the pupils and on the collective discussions of ideas in order to draw conclusions shared by the entire class.

QUALITATIVE ANALYSIS OF GRAPHS

Given the breadth of the research, we shall concentrate here on the results of several experiments, carried out mainly in the second year of the triennium, which concern the recognition and the qualitative interpretation of graphs relating to physical phenomena not associated to formalised rules, for what they express compared to the variability of the magnitudes in question. In particular, we shall look at the main concepts and difficulties encountered by pupils by analysing their work on the co-ordination of the graphic representation of a given phenomenon and the character of the same phenomenon when expressed verbally or which derives from the relative interpretation.

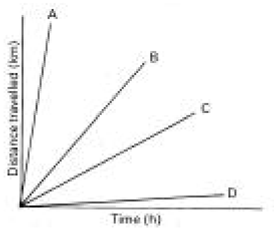
First of all, it must be said that at this scholastic level, the difficulties encountered in analysing the ability to interpret a graph (even from a qualitative point of view) depend on the choice of the phenomenon itself. However simple a physical phenomenon may be to describe formally and however linked it may be to the everyday experience of the pupil, it is often affected by a personal, "naive" interpretation of the phenomenon which is not always faithful to the interpretation of a formalised science (Janvier 1998). A didactic framework is also necessary in this respect so as to lead the interpretation from the actual experience of the pupil to the scientific interpretation. However, in this research, the prospective is inverted: it is the phenomenon which becomes instrumental to the analysis of the difficulties found in the various representative registers (tabular, graphic, algebraic) of a functional correspondence. The examples used are taken from everyday experiences, despite being less easy to formalise in algebraic terms.

The activities that we shall look at here were presented to the pupils together with others on the construction of graphs, based both on tables of data and on verbally described phenomena, like those studied by Chazan & Challis Bethell (1994). Several of these were taken from the NMP project edited by Harper (1987).

A starting task

The first activity which we shall discuss was proposed almost provocatively in the initial phase of the course and is based on the qualitative description of a graph. The aim was to test out the pupils' previous knowledge and their spontaneous, naive ability to interpret a graph. At the same time, we wanted to introduce the topic critically during the following discussion of the various interpretations

Table 2



The four lines in the diagram are the graphs for:
- A croise liner sailing in the Mediterranean
- A camel crossing the Sahara desert
- Concorde travelling at cruising speed across the Atlantic
- The Orient Express in its way to Vienna

The ship, the animal, the aircraft and the train are all travelling at steady speed.

Which graph is which?

given. The motivation behind this choice was the fact that in a prior analysis the relational link in the phenomena in question between time and distance travelled was believed to be closer to the experience of the pupils and easy to interpret by intuition (see table 2).
Furthermore, the comparison between such different types of movement, despite being grouped together in a simplified representation, looked like it might favour the objectification of the relationship between magnitudes. The outcome of the test highlighted comprehension difficulties not previously foreseen like, for example, the scarce familiarity that several of the pupils had with the means of transport considered (Concorde, speed of a ship, etc.). As predicted, the kids' interpretation was guided by the idea of ordering the various speeds (camel, ship, train, plane). However, about 50% of the pupils gave an "inverse" order, coming up with the following association: animal-A; ship-B; train-C; plane-D, which was probably due to the association of the distances on the time axis and to the consequent swap of the axes.

A mid-term task

As has been pointed out elsewhere, physical phenomena of motion generate ambiguity around the rules that describe them. (i.e. trajectory – velocity). This is observable perhaps even more significantly in subjects that approach these problems in a still "naive", intuitive way, without long diatribes on the matter from a non problem-based, more prescriptive style of teaching (as if often the case in secondary school where often such ambiguity persists or is even strengthened if not carefully "disassembled"). In merit of this, the interpretations of the pupils about the task (reported in table 3) - which concerns the fall of a parachutist- are shown. The point of choosing this test is to verify the ability to identify from a given series the graph representative of the variations in altitude in relation to the time taken. The graphs were supported by a table of data as we wanted to look at the role these played in the strategies employed by the pupils.

<u>The behaviour of the pupils</u>

Generally speaking, the behaviour of the pupils can be classified in three categories.

C.1. *Prevalence of the graphical register*. This covers the pupils who make verbal considerations on the phenomenon by trying to connect them to graphics, without making reference to the data (among these, several identify the graph correctly).

C.2. *Co-ordination between the graphic and numerical-tabular register*. This covers the students who start off by analysing the phenomenon qualitatively. They make hypotheses on the possible graphs in relation to the given numeric values, sometimes without proceeding to the drawing of the graph, they simply choose one by the exclusion of the others. (The majority belong to this group).

C.3. *Prevalence of the numerical-tabular register*. This covers the pupils that go on to draw the graph and then compare it to those in the given series. (Only a few weak pupils do this).

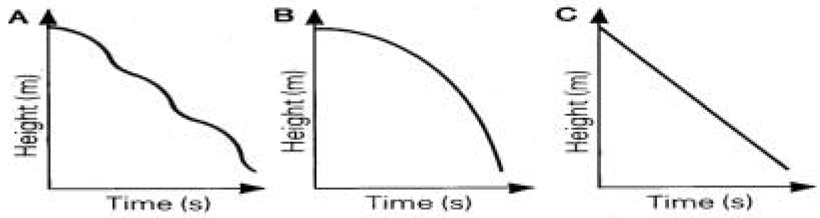
Some of the interpretations given, which are reported below, seem to us to be particularly meaningful when trying to demonstrate difficulties and misconceptions encountered which in a normal classroom environment risk being overlooked.

Table 3

The following table refers to a parachutist's free falling jump from an aeroplane:

time (s)	0	5	10	15	20
height (m)	3000	2875	2500	1875	1000

a) how high is the plane the moment you start the drop?
b) how many meters lower was the parachuter after the first 5 seconds?
c) one of these drops describes the drop. Which? Why?



Choice of the graph

Graph n. 1, though initially chosen intuitively by several pupils, is then excluded in the light of the numeric data. For example, one pupil initially indicates it as a possibility, but as soon as he puts the numeric data onto the axis, he changes idea and writes "No, to be redone". He then draws the polygon and interprets the various parts in relation to the variation of the phenomenon, he comments on the first part of the line drawing an arrow to it and writing "descends slowly". Lastly, he labels the second part with the expression "then starts faster and faster" and the third "in the end he descends very fast.".

Graph n. 2 was chosen by 6 students out of 23. Here are some of the most meaningful justifications: Alberto: "I reckon the distance he travels every five seconds is not the same; in the first bit he goes less far than in the other distances"; Sara B: "It's graph number two because as you can see below (she draws the graph correctly) with the numbers it forms a kind of curve because the speed isn't always the same, it changes." Antonio: "It's the second graph. I get the idea that it's that one because straight after the flat bit there's a great fall." Note that Antonio does not need to draw the graph in order to identify the right one.

Graph n. 3 was chosen by 16 pupils out of 23. Here are some of the most meaningful justifications: Ivan: "…because as the seconds go by it gets faster and also because I think it moves in a straight line."; Vincenzo: "…because it shows all of the fall."; Alessia: "The graph that describes the jump is the third one because when it falls, an object falls very quickly so little by little as the seconds go by the parachutist falls very quickly, he falls almost straight down."; Sara: The right graph is the third one because the parachutist starts from high up and goes down constantly." Ivan, Vincenzo and Alessia evidently read the trajectory of the movement and Sara attributes a constant speed to the fall, whereas other pupils get caught out because they construct the graph directly from the values given in the table, they represent the altitudes wrongly (without respecting the scale), putting all the values at equal distances like the timings and thus obtain a straight line on the graph. Some of them, despite selecting the right graph after considering the phenomenon, when transferring the data onto the graph make mistakes and end up with a straight line which leaves them stuck.

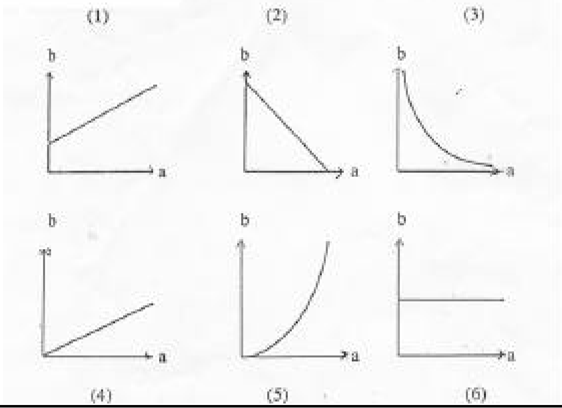
One positive attitude developed by the pupils, common even among those that got it wrong, is shown by how they carry on from the analysis of the phenomenon, going on the strength of their hypotheses in **the** selection of the graph and only then, as a kind of cross-check, going through the graphic representation of the data. With regard to this, it should be noted that higher level pupils do without this extra check on all of the graphs, checking only the one that they have already selected as the right answer. This widespread behaviour can be compared to a similar one typical of this age group in argumentative/demonstrative matters (Malara & Iaderosa 1999) concerning the validity of mathematical or geometric properties – many pupils, while stating their thoughts rationally on the situation in hand, then go on to the numeric cross-check to back up their conclusion.

A final task

The work plan shown in table 4, used in the third year, is used to examine the ability to analyse graphs picking out specific phenomena and the conceptualisation of salient features of their development. The

Table 4

Describe verbally for each of the six following graphs observing how *b* varies when *a* does.
What do you think the difference is between graph 1) and graph 4)?
Try and imagine for graph 6) what might be two possible measurements for *a* and *b*.



pupils' output indicates a widespread difficulty in working abstractly. To be precise, we may note that: i) Only one pupil out of 15 confuses the rising and falling patterns in graphs (1) and (2). ii) One pupil does not recognise rising/falling, but in the case of the curves (graphs (3) – (5)), he indicates graph (4) as falling. iii) Four pupils describe the form of the graph (straight line, parabola etc.) without mentioning the relationship between the two variables. 4i) Only three pupils correctly describe the relationship in graph (6), while another group defines the situation saying that variable *a* and variable *b* vary in the same way. From the explanations given by the kids, it becomes clear that: i) they have difficulty in describing the curve (3) and the straight line (2). (Some say that when *b* diminishes, *a* increases instead of observing how *b* varies when *a* does;); ii) in graph (3), some are struck by the fact that the axes are not touched by the curve, while in the other cases, at least one of the axes is in contact with the graph; iii) they are not able to imagine two measurements to associate with *a* and *b* in the case of (6).

BRIEF CONCLUSIVE CONSIDERATIONS

The results about the pupils' real understanding of the meaning of a given graph, from the observation of its tendency, confirm the foreseen difficulties of reading if not preceded by activities of effective construction. In particular, pupils confuse increasing and diminishing development patterns, especially in the case of non-straight lined graphs. As far as we have been able to observe in the course of their studies, these difficulties persist substantially into the third year of middle school, even when the pupils have been introduced to further graph-drawing activities starting from functions given to them in the form of algebraic formulation.

On the general level, the research carried out highlights the possibility to obtain a higher performance, at least compared to the average in our country, with regard to a qualitative interpretation of graphs in relation to given phenomena. For example, when faced with the problem of interpreting a graph regarding petrol consumption over the period of a hypothetical journey in which there is a straight line parallel to the axis of the abscissa, the pupils give the correct interpretation of that section, saying that it represents a stop. Some difficulties persist however in certain pupils with the interpretation of parts of the graph that correspond to constant values of a measurement represented on axis y (in the case in question, the pupils see the section as a point in which there is a constant speed rather than a stop). For this type of interference of a psychological nature, we would suggest that they tend to exclude the absence of variability of one measurement in the presence of another variation (for example, they tend not to consider the possibility of the absence of movement in certain points in the phenomena of movement). The study of the development of a graph in the absence of a phenomenon of reference and the identification of phenomena and pairs of measurements which correspond in a given graph continue to be problematic for pupils at this scholastic level. In the future we intend to analyse the reasons behind these difficulties in greater depth.

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