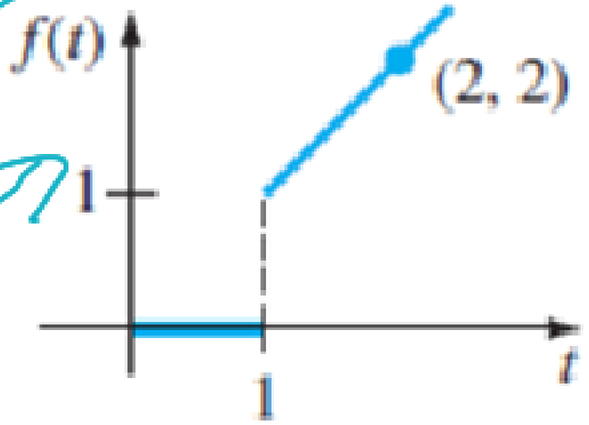
19.2 Transformada de Laplace

Sea $f : [0, +\infty[\to \mathbb{R}$ *y suponga que* $\int_a^{+\infty} e^{-st} f(t)dt$ *converge para ciertos valores de* s*, entonces para dichos valores se define la nueva función* $F(s)$ *que se denomina la Transformada de Laplace de* f*, de la siguiente manera:*

$$\boxed{F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^{+\infty} e^{-st} f(t)dt}$$

▶ Hallar $\mathcal{L}[f(t)]$.



$\rightarrow \qquad f(t) = \begin{cases} 0 & 0 \le t \le 1, \\ t & t > 1. \end{cases}$

Por definición,

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt = \lim_{b\to\infty} \int_1^b t e^{-st} dt = \lim_{b\to\infty} \left(\frac{-be^{-bs} + e^{-s}}{s} + \frac{-e^{-bs} + e^{-s}}{s^2} \right) = \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2} \qquad s > 0.$$

$$= \lim_{b\to+\infty} \left(\int_0^b f(t) e^{-st} dt = \lim_{b\to+\infty} \int_0^1 f(t) e^{-st} dt \to 0 \right.$$

$$+ \int_1^b f(t) e^{-st} dt = \lim_{b\to+\infty} \int_1^b (1) e^{-st} dt$$

otro ejemplo:

Sea $f(t) = e^{at}$

Calcular $\mathcal{L}\{e^{at}\}(s) \overset{\text{definición}}{=} \int_0^{+\infty} e^{at} e^{-st}\, dt$

$$= \int_0^{+\infty} e^{(a-s)t}\, dt = \lim_{b \to +\infty} \int_0^b e^{(a-s)t}\, dt$$

$$= \lim_{b \to +\infty} \frac{1}{a-s} e^{(a-s)t} \Big|_0^b = \lim_{b \to +\infty} \left[\cancel{\frac{1}{a-s} e^{(a-s)b}} - \frac{1}{a-s} \right]$$

$$= -\frac{1}{a-s} = \frac{1}{s-a}$$

* $a - s < 0$
* $a < s$
* $s > a$

$\therefore$ $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$$\lim_{b \to +\infty} e^{(a-s)b} = \lim_{b \to +\infty} \frac{1}{e^{-(a-s)b}}$$

Sup. $\boxed{a - s < 0}$ $= \lim_{b \to +\infty} \frac{1}{e^{\boxed{(s-a)} b}} = 0$

$\boxed{s - a > 0}$

$+$

19.3 Linealidad de la Transformada de Laplace.

Teorema 19.1. Linealidad de la Transformada de Laplace.

Sean

- $f : [0, +\infty[\to \mathbb{R}$, *función con TdL* $F(s)$ *para* $s > a$
- $g : [0, +\infty[\to \mathbb{R}$, *función con TdL* $G(s)$ *para* $s > b$

entonces si $\alpha, \beta \in \mathbb{R}$*:*

$$\boxed{\mathcal{L}\{\alpha f(t) + \beta g(t)\}(s) = \alpha\mathcal{L}\{f(t)\}(s) + \beta\mathcal{L}\{g(t)\}(s) = \alpha F(s) + \beta G(s)}$$

para $s > \max\{a, b\}$.

$$\int f + g = \int f + \int g$$

$$\int a f = a \int f$$

i) $\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

ii) $\mathcal{L}\{\alpha \cdot f(t)\} = \alpha \mathcal{L}\{f(t)\}$

Ejemplo 19.2. Encontrar la TdL de $f(t) = 2e^{-t} + 3\sin(2t)$

Desarrollo: Es claro que, por linealidad de la TdL

$$\mathcal{L}\{2e^{-t} + 3\sin(2t)\}(s) = 2\mathcal{L}\{e^{-t}\}(s) + 3\mathcal{L}\{\sin(2t)\}(s) \tag{19.6}$$

Usando (19.3) y (19.5), se tiene que

$$\begin{aligned}
\mathcal{L}(2e^{-t} + 3\sin(2t))(s) &= 2 \cdot \frac{1}{s+1} + 3 \cdot \frac{2}{s^2+4} \\
&= \frac{2}{s+1} + \frac{6}{s^2+4}
\end{aligned}$$

relación que es válida para $s > \max\{-1, 0\} = 0$.

$$f(t) = 2e^{-t} + 3\,\text{sen}(2t)$$

$$\mathcal{L}\{\underline{f(t)}\} = \mathcal{L}\{2e^{-t} + 3\,\text{sen}(2t)\}$$

Linealidad (1)

$$= \mathcal{L}\{2e^{-t}\} + \mathcal{L}\{3\,\text{sen}(2t)\}$$

Linealidad (2)

$$= 2\,\boxed{\mathcal{L}\{e^{-t}\}} + 3\,\boxed{\mathcal{L}\{\text{sen}\,2t\}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$= 2\,\boxed{\frac{1}{s-\underbrace{(-1)}_{a}}} + 3\,\frac{2}{s^2+2^2}$$

$$= 2 \cdot \frac{1}{s+1} + \frac{6}{s^2+4} = \frac{2}{s+1} + \frac{6}{s^2+4}$$

$$\mathcal{L}\{2e^{-t} + 3\,\text{sen}\,2t\} = \frac{2}{s+1} + \frac{6}{s^2+4} \;//$$

Listado de Transformadas de Laplace de funciones especiales

	$f(t)$	$F(s)$
(1)	$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$
(1a)	1	$\frac{1}{s}$
(1b)	t	$\frac{1}{s^2}$
(2)	e^{at}	$\frac{1}{s-a}$
(2b)	$e^{at}t^n,\ n \in \mathbb{N}$	$\frac{n!}{(s-a)^{n+1}}$
(3)	$\sin(at)$	$\frac{a}{s^2+a^2}$
(4)	$\cos(at)$	$\frac{s}{s^2+a^2}$
(5)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
(6)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
(8)	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
(9)	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
(10)	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
(11)	$e^{at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$

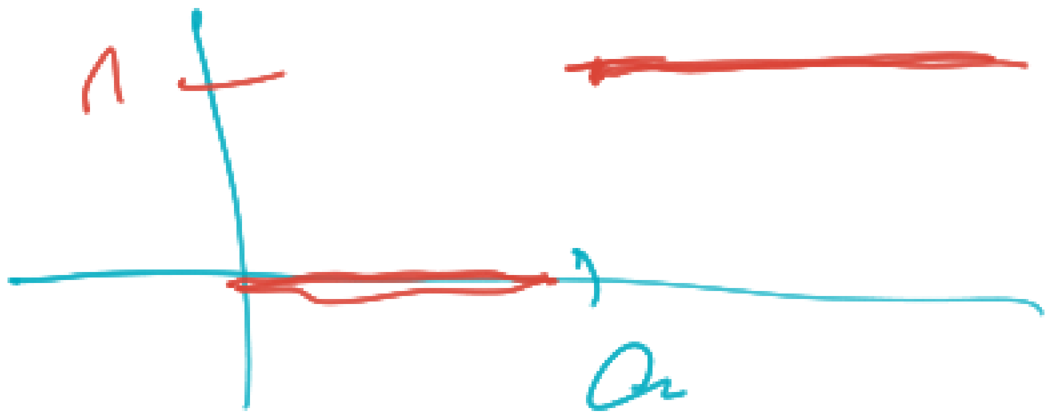
Propiedades generales de la Transformada de Laplace

Definición	$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^{+\infty} e^{-st} f(t)dt$
Linealidad	$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$
Dilatación	$\mathcal{L}(f(at)) = \frac{1}{a}F(\frac{s}{a}), \quad a > 0$
Derivación	$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$
	$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$
	$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$
Integración	$\mathcal{L}\{\int_0^t f(u)u\} = \dfrac{1}{s}\mathcal{L}\{f(t)\}$
Dualidad	$\mathcal{L}\{t\, f(t)\} = -F'(s)$
	$\mathcal{L}\{t^n\, f(t)\} = (-1)^n F^{(n)}(s)$
Traslación en s	$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$
Traslación en t (1)	$\mathcal{L}\{f(t-a)\cdot u(t-a)\} = e^{-as}F(s)$
Traslación en t (2)	$\mathcal{L}\{f(t)\cdot u(t-a)\} = e^{-as}F(s+a)$
	$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$
	$u(t-a) = \begin{cases} 0 & \text{si} \quad t < a \\ 1 & \text{si} \quad t \geq a \end{cases}$

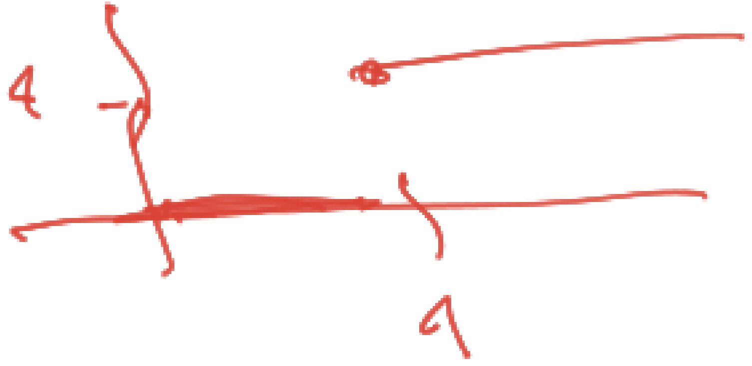
Usando Formulario de T de L, hallar

$$\mathcal{L}\{f(t)\} \text{ donde } f(t) = \begin{cases} 0 & \text{si } 0 \le t < 1 \\ t & \text{si } t \ge 1 \end{cases}$$

Nota: Función escalón unitario (o función de Heaviside)



$u(t-a)$



$$\boxed{f(t) = t\,u(t-1)}$$

$$f(t) = \underbrace{[(t-1)+1]}_{f(t-1)}\cdot u(t-1)$$

$$f(t) = t+1$$

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$$

$$f(t) = \begin{cases} 0 & 0 \le t \le \\ t, & t > 1 \end{cases}$$

$$\mathcal{L}\{f(t-1)\,u(t-1)\} = e^{-s}\,\mathcal{L}\{f(t)\}$$

$$= e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

donde

$$f(t) = t + 1$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t + 1\} = \mathcal{L}\{t\} + \mathcal{L}\{1\} = \frac{1}{s^2} + \frac{1}{s}$$

$$f(t) = t^2 e^{-3t}$$

$\mathcal{L}\{t^2 e^{-3t}\} =$

Fórmula 2b

$\mathcal{L}\{e^{at} t^n\} = \dfrac{n!}{(s-a)^{n+1}}$

$\mathcal{L}\{e^{-3t} t^2\} = \dfrac{2!}{(s-(-3))^3} = \dfrac{2}{(s+3)^3}$

$a = -3$

$n = 2$

d) $f(t) = e^{-4t}(t^2+1)^2$

$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$\mathcal{L}\{e^{-4t}(t^2+1)^2\} = \frac{4!}{(s+4)^5} + \frac{4}{(s+4)^3} + \frac{1}{s+4}$

$f(t) = (t^2+1)^2$

$F(s) = \mathcal{L}\{f(t)\} =$

$= \mathcal{L}\{(t^2+1)^2\}$

$= \mathcal{L}\{t^4+2t^2+1\}$

$= \mathcal{L}\{t^4\} + 2\mathcal{L}\{t^2\} + \mathcal{L}\{1\}$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$= \frac{4!}{s^5} + 2\cdot\frac{2!}{s^3} + \frac{1}{s} \longrightarrow F(s)$