Resolver, usando matrices:

$$\vec{X}' = \underbrace{\begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & 6 \\ -4 & 0 & -3 \end{pmatrix}}_{A} \vec{X} \qquad \vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \vec{X}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

PASO 1: Buscar V.P. de la matriz A

(Soluciones de la ec. característica)

$$* \quad |A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 2 \\ 3 & -\lambda & 6 \\ -4 & 0 & -3-\lambda \end{vmatrix} = 0 \Rightarrow \begin{array}{l} \lambda_1 = \boxed{-3} \longrightarrow \text{v.p?} \\ \lambda_2 = \boxed{1+2i} \longrightarrow \text{v?} \\ \lambda_3 = 1-2i \end{array}$$

V.P. $\longrightarrow \lambda_1 = -3$

$$\begin{pmatrix} 5 & 1 & 2 \\ 3 & 3 & 6 \\ -4 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\boxed{\text{V.P.}}$

$$\left.\begin{array}{ll} (1) & 5a + b + 2c = 0 \\ (2) & 3a + 3b + 6c = 0 \\ (3) & -4a = 0 \end{array}\right\} \rightarrow \boxed{a = 0}$$

Reemplazando $a=0$ en (1) y (2)

$$\left.\begin{array}{l} b + 2c = 0 \\ \cancel{3b + 6c = 0} \end{array}\right| \quad \begin{array}{l} b + 2c = 0 \\ c = 1 \Rightarrow b = -2 \end{array}$$

∴ El vector propio, asociado al $\lambda_1 = -3$ es:

$$V_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

Luego:

$$Y_1 = C_1 e^{-3t} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

* V.P. asociado a

$\lambda_2 = 1+2i$

$$\underbrace{(A - \lambda_2 I)}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-2i & 1 & 2 \\ 3 & -5-2i & 6 \\ -4 & 0 & -4-2i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left\{\begin{array}{lll} ① & (1-2i)a + b + 2c = 0 \\ ② & 3a - (5+2i)b + 6c = 0 \\ ③ & -4a - (4+2i)c = 0 \end{array}\right.$$

$$\textcircled{1}\quad (1-2i)a + b + 2c = 0$$
$$\textcircled{2}\quad 3a - (1+2i)b + 6c = 0$$
$$\textcircled{3}\quad -4a - (4+2i)c = 0$$

$$\textcircled{2} \times 4: \quad \cancel{12a} - (4+8i)b + 24c = 0$$
$$\textcircled{3} \times 3: \quad -\cancel{12a} - (12+6i)c = 0$$
$$+ \quad -(4+8i)b + (12-6i)c = 0$$
$$\boxed{-(2+4i)b + (6-3i)c = 0}$$

$$\begin{aligned}
&① \quad (1-2i)a + b + 2c = 0\\
&② \quad 3a - (1+2i)b + 6c = 0\\
&③ \quad -4a - (4+2i)c = 0
\end{aligned}$$

$$\begin{aligned}
&① \times (-3) : \quad (-3+6i)a - 3b - 6c = 0\\
&② \times (1-2i) : \quad (3-6i)a - 5b + (6-12i)c = 0
\end{aligned}$$

$$-8b - 12ic = 0$$

$(*_1)$ $\quad +2b + 3ic = 0$

$(*_2)$ $\quad -(2+4i)b + (6-3i)c = 0$

$(*_1)$: $\boxed{b = -\frac{3}{2}ic}$ $\xrightarrow{(*_2)}$ $+(2+4i)\frac{3}{2}ic + (6-3i)c = 0$

$$\cancel{3ic} - \cancel{6c} + \cancel{6c} - \cancel{3ic} = 0$$

$$0 = 0$$

$$\boxed{c = i}$$

1) $(1-2i)a + b + 2c = 0$

$$2b + 3ic = 0 \;\longrightarrow\; 2b + 3i \cdot i = 0$$

$$2b - 3 = 0$$

$$\boxed{b = \frac{3}{2}}$$

$$(1-2i)a + \frac{3}{2} + 2i = 0 \quad | \cdot 2$$

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$$(2-4i)a + 3 + 4i = 0$$

$$(2-4i)a = -(3+4i)$$

$$a = -\frac{3+4i}{2-4i} \cdot \frac{2+4i}{2+4i} = -\frac{-10+20i}{20}$$

$$= +\frac{1}{2} - i$$

$$\boxed{a = \frac{1}{2} - i}$$

o: El vector propio asociado a $1+2i$ es

$$\vec{W}_1(t) = e^{\alpha t}[\vec{U}\cos(\beta t) - \vec{V}\sin(\beta t)]$$

$$\vec{W}_2(t) = e^{\alpha t}[\vec{U}\sin(\beta t) + \vec{V}\cos(\beta t)]$$

$1+2i \to \alpha \pm \beta i$

$$\begin{array}{l} a \to \\ b \to \\ c \to \end{array} \begin{pmatrix} \frac{1}{2} - i \\ \frac{3}{2} \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} 1/2 \\ 3/2 \\ 0 \end{pmatrix}}_{U} + i \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{V}$$

$$Y_2 = e^{t}\left[\begin{pmatrix} 1/2 \\ 3/2 \\ 0 \end{pmatrix}\cos(2t) - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\operatorname{sen}(2t)\right]$$

$$Y_3 = e^{t}\left[\begin{pmatrix} 1/2 \\ 3/2 \\ 0 \end{pmatrix}\operatorname{sen}(2t) + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\cos(2t)\right]$$

$$\vec{Y}_2 = e^{t}\left[\begin{pmatrix}1/2\\3/2\\0\end{pmatrix}\cos(2t) - \begin{pmatrix}-1\\0\\1\end{pmatrix}\operatorname{sen}(2t)\right]$$

$$\vec{Y}_3 = e^{t}\left[\begin{pmatrix}1/2\\3/2\\0\end{pmatrix}\operatorname{sen}(2t) + \begin{pmatrix}-1\\0\\1\end{pmatrix}\cos(2t)\right]$$

Finalmente, la solución general del SEA trabajado es:

$$\vec{Y} = c_1 e^{-3t}\begin{pmatrix}0\\-2\\1\end{pmatrix} + c_2 \vec{Y}_2 + c_3 \vec{Y}_3$$

$$\vec{Y}(0) = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\-2c_1\\c_1\end{pmatrix} + c_2\begin{pmatrix}1/2\\3/2\\0\end{pmatrix} + c_3\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$x(0)=0$

$y(0)=0$

$z(0)=1$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\-2c_1\\c_1\end{pmatrix} + c_2\begin{pmatrix}1/2\\3/2\\0\end{pmatrix} + c_3\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

de donde

$$\begin{aligned}
&(1)\quad 0 = \tfrac{1}{2}c_2 - c_3\\
&(2)\quad 0 = -2c_1 + \tfrac{3}{2}c_2\\
&(3)\quad 1 = c_1 + c_3
\end{aligned}$$

(1) + (3):

$$\begin{aligned}
&(4)\quad c_1 + \tfrac{1}{2}c_2 = 1 \quad \cdot 2\\
& -2c_1 + \tfrac{3}{2}c_2 = 0
\end{aligned}$$

$$\begin{aligned}
2c_1 + c_2 &= 2\\
-2c_1 + \tfrac{3}{2}c_2 &= 0\\
\hline
\tfrac{5}{2}c_2 &= 2
\end{aligned}$$

$$\boxed{c_2 = \tfrac{4}{5}}$$

en (4) $c_1 + \frac{1}{2}c_2 = 1$

$c_1 + \frac{1}{2}\cdot\frac{4}{5} = 1$

$$\boxed{c_1 = 1 - \tfrac{2}{5} = \tfrac{3}{5}}$$

en (3): $1 = \frac{3}{5} + c_3$

$$\boxed{c_3 = 2/5}$$