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Does a cube have an equation?

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Abstract

This article looks at the question of whether and how a geometrical cube may be determined as the solution set of a single equation. Beyond this being an interesting and surprising result for upper high school and first year college students, we argue that, as an investigative activity, it has value in refining students' images of what solutions of equations can be and, with that, refining their ideas of what an equation is.

1. Introduction

1.1. Images of cubes and equations

Before we ask 'Does a cube have an equation?' we might first ask 'What is a cube?' The two questions are obviously related, for to ask what is a cube is to ask whether it is the sort of thing that has an equation; alternatively, of course, we can ask whether an algebraic equation is the sort of thing that describes a cube. But first, what sort of thing is a *cube*? More precisely, what images might the word 'cube' evoke in the mind of a high school or beginning college student?

Images related to mathematics—though not quite mathematical definitions—include the square-faced regular solid or its surface and the third power of a variable, 'x-cubed'. But even mathematically minded students are not mathematicians 24 h-a-day; they are exposed to the word 'cube' in many other non-mathematical contexts in their everyday lives. A quick internet search brings up the following:

Cube (film) — is 1997 a Canadian science fiction movie
The Cube — is a 1969 movie by Jim Henson.
Nissan Cube — a type of car.
Nintendo Game Cube, often referred to as *cube* for short.
Cube (game) — a free software first-person shooter
Power Mac G4 Cube — a compact computer from Apple.
Cube Microplex — a cinema in Bristol
Cube, the butler—a character in the simulation game series *Princess Maker*
OLAP cube—a representation of relational data used in Data Warehousing

The connotations are clear: a cube is a solid thing, just like the butler in the *Princess Maker*; it is a structure—the frame of the cube—like the OLAP cube; it is a certain snug box-shaped space, like the *Nissan Cube*. Incidentally, a second model of the *Nissan Cube*, the *Cube 3*, is advertised as the *Cube*^{3}, just like x^3.

These are not, as we have said, mathematical contexts. But the associations created by such uses of the word 'cube' may contribute, at least, to students' sense of what a cube is when they do think about mathematics. The importance of images such as these for students' understanding of mathematical concepts has been noted in the many studies on analogies and visualization in mathematics education [e.g. (1)]. Vinner's and Tall's notion of a *concept image* (2,3) also calls attention to these non-mathematical associations, stressing how such associations and images—rather than strict mathematical definitions (even when students know those definitions)—are often what are truly at work in students' mathematical thinking.

And concept images apply to equations no less than to cubes. Asked what sort of objects are determined by equations, it is likely that students will think of straight lines or any one of several smooth curves; if they have some experience with equations in several variables, they might also think of a smooth surface: a plane or the surface of a sphere. A solid polyhedral, like a cube, or a surface made up of six planes, or a frame of 12 edges are not usually associated with objects determined by single algebraic equations.

Breaking away from such associations is a first and crucial step towards a mathematically precise approach to ideas such as equations and cubes. Thus, if our goal is in fact to prepare students for advanced mathematical work or for fields that require higher mathematical thinking, Vinner suggests that '... one should do more than introduction the definition. One should point at the conflicts between the concept image and the formal definition and deeply discuss the weird examples (like the tangent to the graph of $y = x^3$ at (0,0) or the limit of the sequence whose n-th element is $(-1)^{2n}$, $n = 1,2,3, \ldots$,etc.)' [(3), p.80].

This then brings us back to our initial question: Does a cube have an equation? The question by itself immediately challenges the usual sense of what an equation can describe and makes one wonder what a cube really is if it has an equation. As a mathematical task, finding an equation for a cube fits well what Kieran (4) has called a 'generational' algebraic activity, an activity in which the student must provide an algebraic representation for a mathematical object; but because a cube is, as Vinner puts it, a 'weird' example, it goes beyond an ordinary 'generational activity' and forces the students to confront the very meaning of an algebraic representation.

With such ends in mind, we wish to demonstrate in this article that a cube does indeed have an equation, and, that, in more than one way. Indeed, we shall show that a cube has an equation regardless of whether one takes the cube to be its volume, its faces, the frame of its edges or its vertices!

2. Equations of cubes

2.1. Equations relying on domains of definition

To start, consider the following equation:

$$\sqrt{1-x^2} \cdot \sqrt{1-y^2} \cdot \sqrt{1-z^2} = 0 \tag{1}$$

We asked several groups of first-year students at an engineering college what they thought the solution or solutions to this equation might be. In these open discussions, the responses that arose most often were: a point, two points, eight points and a sphere. Those who thought a single point solved the equation fixed on the point (1,1,1); those who thought the equation had two points as its solution recognized that, besides (1,1,1), also $(-1,-1,-1)$ solved the equation. Other students, realizing that it was unnecessary for all the coordinates to be 1 or -1, said that there were eight solutions, that is, all possible points of the form $(\pm1,\pm1,\pm1)$. As a response, 'a sphere', was the most interesting of course: the students seemed to be reacting to a stretched resemblance between the given equation and the equation $1 - x^2 - y^2 - z^2 = 0$.

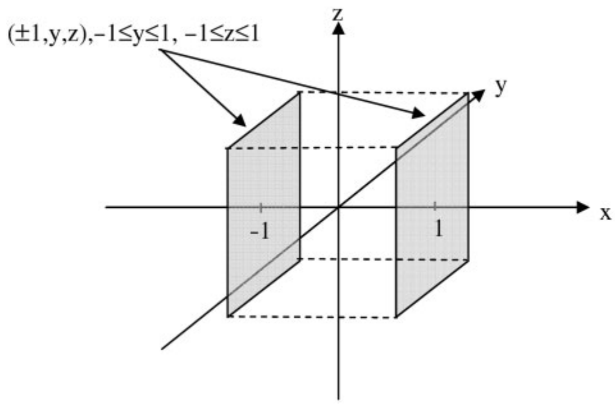

Fig 1. Faces of the cube normal to the x-axis.

But both where the students suggested certain points as the solutions of the equation and where they suggested a sphere, they were responding with objects that they had seen before as solutions of equations. Or to use the terminology in our introduction, such solutions fit their image of a solution of an equation—one variable or many. Hardly, any of the students, without generous hints, gave the correct answer: the surface of a cube.

Yet they really were quite close to the answer. For in recognizing that all possible points of the form (±1,±1,±1) they were beginning to recognize that, because the equation is reducible, each factor can be treated separately from the others and each is an expression in one variable. Thus, finding a zero, x_0, for, say, the first factor will provide a set of zeros for the entire equation, namely, all points of the form (x_0,y,z)—that is, *all points for which the numbers y and z are within the domain of the second and third expressions*. The proviso at the end is sometimes viewed by students as a mere technical matter; however, it is because of that restriction that the solutions of (1) are bounded, that they are, in fact, the faces of the cube. For the second and third square root expressions are defined only for $-1 \leqslant y \leqslant 1$ and $-1 \leqslant z \leqslant 1$, respectively. Thus, one set of solutions of equation (1) is $(\pm1,y,z)$, with $-1 \leqslant y \leqslant 1$ and $-1 \leqslant z \leqslant 1$, which represent the two faces of a cube normal to the x-axis (Fig. 1).

Altogether, then, the solution set for (1) is three pairs of square faces normal to the x, y and z-axes:

$$(\pm1, y, z), \quad -1 \leqslant y \leqslant 1 \quad \text{and} \quad -1 \leqslant z \leqslant 1$$

$$(x, \pm1, z), \quad -1 \leqslant x \leqslant 1 \quad \text{and} \quad -1 \leqslant z \leqslant 1$$

$$(x, y, \pm1), \quad -1 \leqslant x \leqslant 1 \quad \text{and} \quad -1 \leqslant y \leqslant 1$$

Once one sees in this way how the domain of definition can determine the geometrical character of the solutions of an equation, one is less likely to overlook the effect of multiplying equation (1) by 0:

$$0 \times \left(\sqrt{1-x^2} \cdot \sqrt{1-y^2} \cdot \sqrt{1-z^2}\right) = 0 \tag{2}$$

Multiplying by 0 gives us an equation whose solutions must include all points (x,y,z) for which the expressions are defined. In other words, the solution set of this equation is identical to its domain of definition. This of course, in the case of equation (2), is just $\{(x,y,z)|-1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1, -1 \leqslant z \leqslant 1\}$. The same result can be obtained also by simply equating the left hand side of equation (1) to itself:

$$\left(\sqrt{1-x^2}\cdot\sqrt{1-y^2}\cdot\sqrt{1-z^2}\right) = \left(\sqrt{1-x^2}\sqrt{1-y^2}\sqrt{1-z^2}\right) \tag{2a}$$

Hence, in either case, equation (2), or (2a), is an equation for the solid cube.

Further considerations of the domain of definition, and equation (2a), allow us to write this equation for the cube interior:

$$\frac{\sqrt{1-x^2}\cdot\sqrt{1-y^2}\cdot\sqrt{1-z^2}}{\sqrt{1-x^2}\cdot\sqrt{1-y^2}\cdot\sqrt{1-z^2}} = 1 \tag{3}$$

For again, because this equation is an identity for all points (x,y,z) in domain of definition, the solution set for the equation precisely in that domain: $\{(x,y,z)|-1 < x < 1, \ -1 < y < 1, -1 < z < 1\}$.

If the three square root expressions in equation (1) are added instead of multiplied, we obtain an equation which will yield a solution only if all the expressions vanish simultaneously:

$$\sqrt{1-x^2}+\sqrt{1-y^2}+\sqrt{1-z^2} = 0 \tag{4}$$

For this equation, then, the students' incorrect answer for equation (1) that the solution set comprises the eight points $(x,y,z) = (\pm1,\pm1,\pm1)$ is precisely the right one. So, equation (4) is an equation for the eight vertices of a cube.

What about the frame of the cube, that is, just the 12 edges and eight vertices? Does this have an equation as well? We might begin by looking at the equation given by just two terms in equation (4), for example,

$$\sqrt{1-x^2}+\sqrt{1-y^2} = 0 \tag{5a}$$

Here, z is free, while x and y must be equal to ±1, that is, the solutions are the four lines $(1,1,z)$, $(1,-1,z)$, $(-1,1,z)$, $(-1,-1,z)$, which are the extensions of four edges of a cube. In order to make them the edges of the cube, z must be restricted between -1 and 1. This can be done by multiplying the equation by $\sqrt{1-z^2}$:

$$\left(\sqrt{1-x^2}+\sqrt{1-y^2}\right)\cdot\sqrt{1-z^2} = 0 \tag{5b}$$

However, this is not enough, for when $z = \pm1$, x and y can take on any value such that $-1 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant 1$, as in the case of equation (1). In other words, equation (5b) is the equation of the figure made up of the four edges of the cube parallel to the z-axis and the two faces normal to the z-axis. It looks like we need to find a way to combine somehow the lines we obtain in equation (5a) with the restriction we obtain in (5b). But, in fact, it is sufficient to focus just on the lines. Since (5a) provides us with four extended edges of the cube parallel to the z-axis through, let us consider analogous equations for extended edges parallel to the y and x axes. We obtain the restriction on the length of the edges, by multiplying all three equations:

$$\left(\sqrt{1-x^2}+\sqrt{1-y^2}\right)\left(\sqrt{1-x^2}+\sqrt{1-z^2}\right)\left(\sqrt{1-y^2}+\sqrt{1-z^2}\right)=0 \quad (5c)$$

Equation (5c) can be understood in the following way. Each factor of the left-hand side provides four parallel extended edges of the cube, once in the direction of the x-axis, once in the direction of the y-axis and once in the direction of the z-axis; however, because the equation contains all three expressions $\sqrt{1-x^2}, \sqrt{1-y^2}, \sqrt{1-z^2}$, the domain of the equation—and, therefore, the extent of each set of parallel lines—is restricted to the region of the cube. Hence, the solutions of equation (5c) are precisely all the points along the frame of the cube, as was required.

2.2. A more natural equation

Of course the equations given above for the various views of a cube are not the only such equations. For example, we can easily provide a simple alternative to equation (1) for the surface of the cube:

$$\max(|x|, |y|, |z|) = 1 \quad (6)$$

It is easy to see that the solution set of this equation is precisely the same as that of equation (1), for indeed that is what we get when we state (6) in other terms:

$$\{(\pm 1, y, z) | -1 \leqslant y \leqslant 1, -1 \leqslant z \leqslant 1\}$$

$$\cup\{(x, \pm 1, z) | -1 \leqslant x \leqslant 1, -1 \leqslant z \leqslant 1\}$$

$$\cup\{(x, y, \pm 1) | -1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1\}$$

But this equation, like those in the last section, might leave students with a feeling that they have been tricked, or, at best, that the equation is contrived and does not really exhibit the ‘cubeness’ of the cube. Other equations which typically come into students’ purview more directly reflect some property of the solution set as a geometrical object—the equation of the line, the constancy of its slope; the equation of a circle, the constancy of the distance to some fixed point. Does the cube then have an equation like those, an equation that derives from some essential geometrically intuitive property of a cube?

To answer this, let us begin with a one-dimensional solid ‘cube’, an interval. Consider the interval $[-1,1]$. What does it mean for a point to be within this interval? It means naturally that the point lies *between* -1 and 1. What does it mean for a point x to lie between *any* two points a and b? A point x lies between two given points a and b if and only if distance from a to x together with the distance from x to b equals the distance from a to b [(5), pp. 289–291]. Hence, if $d(x,y)$ is the distance from x to y, then x lies between two points a and b whenever $d(a,x) + d(x,b) = d(a,b)$, or using the absolute value of the difference as a measure of the distance: $|x-a| + |x-b| = |b-a|$ (Fig. 2).

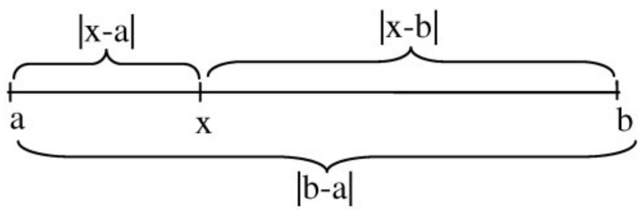

Fig 2. Point x falls between a and b if and only if $|x-a|+|x-b|=|b-a|$.

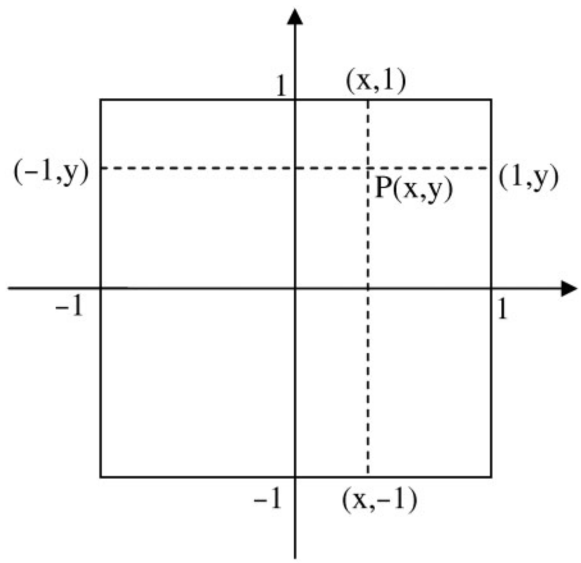

Fig 3. P is in the square if and only if $|x+1|+|x-1|+|y+1|+|y-1|=4$.

For the interval $[-1,1]$, therefore, we can say that a point x lies in the interval if and only if $|x+1|+|x-1| = 2$. So, this is an equation of the interval $[-1,1]$, and it is one expressing the very basic geometrical property of betweenness.

What about a square in the plane, the two-dimensional solid cube? In the case of the interval, we could say that the sum of the distances from the point x to the 'sides' of the interval were equal to the 'width' of the interval. So, by analogy, the key relation for the square will be this: that the sum of the distances from a point $P(x,y)$ in the square to the sides of the square will always equal the sum of the 'width' and 'height' of the square, or twice the length of its side (Fig. 3).

So, again using the absolute value for the distances, we can write the equation of the square: $(|x+1|+|x-1|)+(|y+1|+|y-1|) = 2+2$ or,

$$|x+1|+|x-1|+|y+1|+|y-1| = 4 \tag{7}$$

For the solid three-dimensional cube, we can use then the same reasoning, namely, that any point $P(x,y,z)$ in the cube if the sum of the distances from P to the sides of the cube add up to the

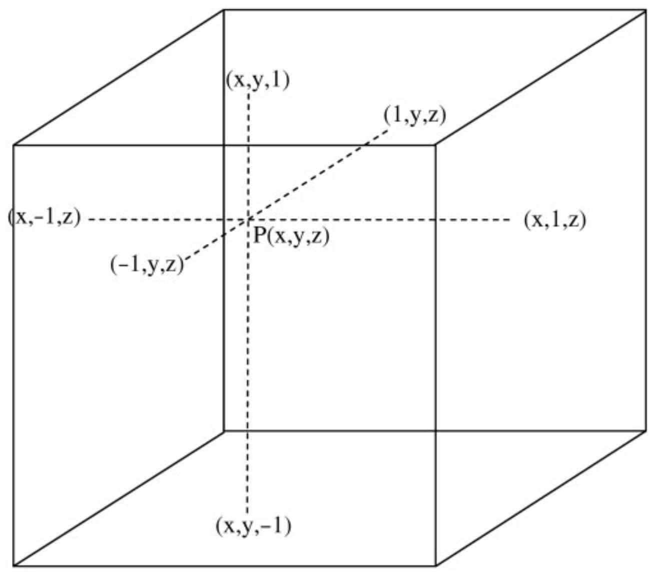

Fig 4. *P* is in the cube if and only if $|x+1|+|x-1|+|y+1|+|y-1|+|z+1|+|z-1|=6$.

sum of the width, length and height of the cube (Fig. 4), which in the special case we have be referring to throughout this paper is 6. Therefore, the equation of the solid cube is:

$$|x+1|+|x-1|+|y+1|+|y-1|+|z+1|+|z-1|=6 \tag{8}$$

Equation (8), although not strictly an algebraic equation, is, nevertheless, an equation making use only of function familiar to upper high school and first year college students. More importantly though, equation (8) is an equation derivable directly from the geometry of the cube, so that, in that sense, it is a *natural* equation for the solid cube.

3. Conclusion

In all the examples above, we found equations for a particular cube, namely, the one whose vertices are the eight points $(\pm1,\pm1,\pm1)$. One might think it would have been more enlightening had we found equations for a more general cube, one whose vertices were the points $(a\pm k, b\pm k, c\pm k)$. But this would be missing the point. Our immediate goal was to show only that a cube is an object which can be defined by an equation. This was done not because we believe it is particularly useful to have an equation of a cube; if that were the case it would no doubt be essential to find an equation for a cube in the most general terms. Indeed, defining a cube by a system of inequalities is probably a more *useful* mode of defining a cube. We asked whether a cube has an equation, and demonstrated that it does, however, as an example of how students can be brought to the question of what an equation is, what a domain of definition is and what sort of geometrical objects equations can determine. The cube is not unique for this purpose (see, for example, 6,7), but a cube is a 'weird' and surprising enough example to challenge students' images of what kinds of objects come with the range of single equations.

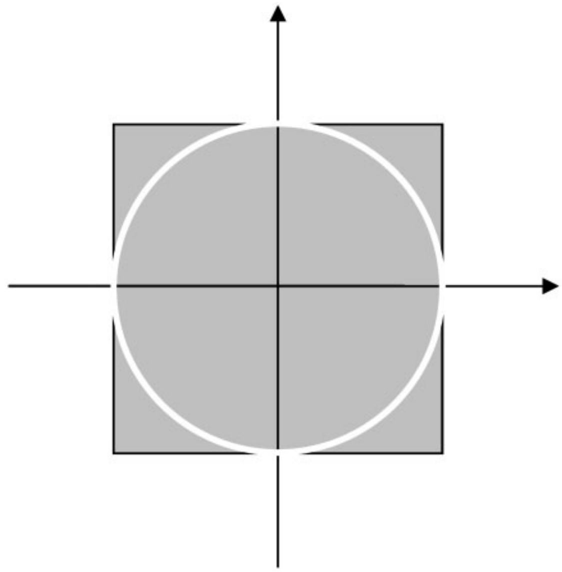

Fig 5. Solution set for $(|x+1|+|x-1|+|y+1|+|y-1|-4)/(x^2+y^2-1)=0$.

The development presented here, we believe, is also a good context to bring out other sorts of mathematical issues. It is standard fare in school mathematics, for instance, to teach that if an equation in one variable is reducible then the complete set of zeros is the union of the zeros of the factors; however, it is rarely pointed out to upper level high school or first year college students how this fact extends to equations of more than one variable, namely, that equations for the unions of curves can be obtained by multiplying their equations. The solution set of $(y-x)(x^2+y^2-1)=0$, for example, is a circle with a line passing through its centre. Moreover, it is generally shown that by dividing an equation whose solution set is T by another equation whose solution set is S, one obtains an equation whose solution set is $T-S$. But again, the meaning of this for equations with more than one variable is not often made concrete, for example, by showing that the solution set for the equation $(|x+1|+|x-1|+|y+1|+|y-1|-4)/(x^2+y^2-1)=0$ is the square area whose vertices are $(\pm1,\pm1)$ minus the unit circle (Fig. 5).

The kinds of issues discussed in this article, then, not only challenge students undeveloped images of algebraic equations, but also can provide some initial ideas, and, yes, images, for students who may later study more advanced topics in mathematics such as algebraic geometry.

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