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Visualization Enhanced by Technology

In the Learning of Multivariable Calculus

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Abstract

Visualization is the process of using geometrical illustrations of mathematical concepts. Over the past centuries, a decline of the geometric element in mathematical instruction has been observed. Computers have given mathematics the opportunity to adjust the balance between the various modes that constitute the basis of mathematical education: the symbolic, the visual, and the numeric. Visualization in multivariable calculus cannot be overemphasized. This paper is the outcome of observations compiled from a multivariable calculus class where computers were used extensively. Results show that visualization skills can be improved and that students use technology in very innovative ways.

Visualization Enhanced by Technology

In the Learning of Multivariable Calculus

For the past decade, many calculus instructors at the college level have been engaged in re-designing their classes to conform to the new ideas and principles that govern the reformed calculus curriculum; a curriculum that emphasizes three components in the teaching and learning of mathematics: the algebraic, the geometric, and the numeric. Until the recent past, the teaching of mathematics was experiencing a steady decline in the geometric and numeric components. Research has shown that the more successful students in advanced mathematics are those that have been exposed to multiple representations of mathematical ideas and principles (see for instance Robert & Boschet (1984)). In practice, however, implementation was lagging. It was not until the advancement of computers and computer graphics that the balance began to adjust between the various modes of mathematical learning. According to Tall (1991, p. 15), the computer "gives mathematical education the opportunity to adjust the balance between various modes of communication and thought that have previously been biased toward the symbolic and sequential". Consequently, the incorporation of the visual and numerical modes in the teaching process is now on the rise.

Visualization is the process of using geometry to illustrate mathematical concepts. According to Zimmerman (1991, p.136), "the role of visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject." In general, proficiency in visual thinking requires from the student an ability to understand that algebra and geometry are alternative ways for expressing mathematical ideas, and that a graph may contain information needed for a better overall picture of a problem.

Research papers assessing the role of visualization in topics that are studied in single variable calculus are numerous. In multivariable calculus, visualization cannot be overemphasized; yet the literature on the subject is still minimal. This paper is the product of observations gathered from a combined multivariable calculus and differential equations course offered at a university located in the northeastern part of the United States. The instructor of the section observed emphasized strongly the visual approach with almost every concept covered in the entire semester. Also, interactive computer graphics programs were used to enhance the learning, particularly the visual aspect of the course.

The Course and the Data

The class observed is a 4-credit course that is not aimed in general at students majoring in mathematics, physics, computer science or engineering. Rather, students' majors may vary between biology, economics, chemistry, meteorology, and others. Such disciplines require a good knowledge of calculus, particularly multivariable. The first half of the course usually covers the multivariable part. The book used was Multivariable Calculus, Concepts and Contexts by Stewart (1998). The instructor met with the students in 50-minute sessions: twice a week in a regular classroom and once in a computer laboratory. Recitation sessions were hold held twice a week by a teaching assistant. Three types of homework were assigned weekly: problems required for the algebraic understanding of the material, applications problems, and visualization problems requiring the use of the computer. The main computer programs used in the lab were 3D-Analyzer and Surfaces, two non-commercial interactive specialized graphics programs written by Marc Parmet.

The section observed consisted of 26 students. The data collected for the study included classroom observations, observations from lab sessions, copies of students' visualization assignments, and copies of their exams. Furthermore, and throughout the semester, questionnaires were being administered assessing the students' visualization skills, and short essays were regularly assigned in which students were to express their thoughts about a specific project or topic that falls under the visual component of the course. Additional data were also collected from interviews conducted with nine students of the class.

Sample Solutions of Visualization Assignments

As mentioned above, the students were assigned weekly exercises emphasizing the visual aspect of the course and requiring in most cases the use of the computer. In what follow, I will present some of the students' work (in chronological order) and will assess the extent to which a visual method of solution was employed. According to N. Presmeg (1989), a solution is considered visual if an essential part of it involves visual imagery, even if reasoning or algebraic methods are also incorporated. The role of technology in enhancing visualization skills will also be evaluated.

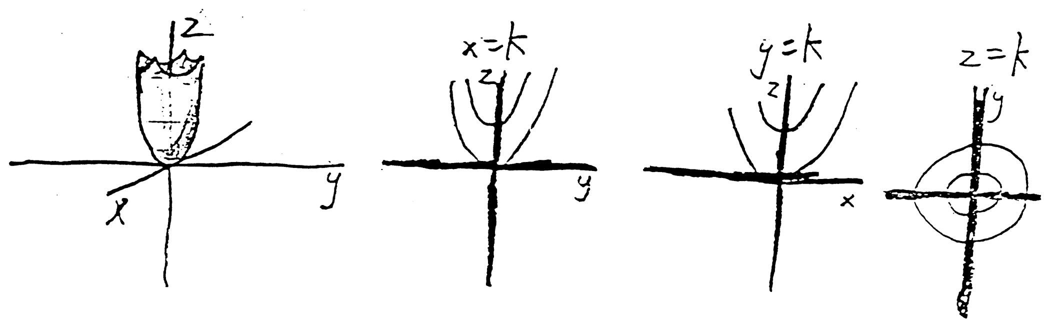
Example 1: Exploring Quadric Surfaces

This visualization assignment asked the students to identify the quadric surface $z = x^2 + cy^2$, for various values of c, and to find or estimate the critical values of c when the shape of the surface changes. The exercise therefore is a visual exploration of the effect of the parameter c on the shape of the surface. The work of two students John and Jim will be assessed.

> John: "This graph always yields a paraboloid, either hyperbolic or elliptic. When c is positive, the graph resembles a cup shape, with the z-axis inside the cup, normal with the bottom. Traces in z = k are ellipses; traces in x = k and y = k are parabolas."

Illustrations by hand-drawn pictures were presented (see Figure 1).

Figure 1. John's Exploration of the surface $z = x^2 + cy^2$, for $c > 0$



John wrote further:

> John: "This is the basic shape for all values c>0. Zero is a critical number, since at zero, the graph changes to a parabolically curved strip. Traces in z = k are pairs of lines, traces in x = k are lines, and

traces in y = k are parabolas." (See Figure 2). "For all values of c<0, the graph has a hyperbolic paraboloid shape. Traces in z = k are hyperbolas, and traces in x = k and y = k are parabolas. The graph resembles a saddle." (See Figure 3)

Figure 2. John's Exploration of the surface $z = x^2 + cy^2$, for $c = 0$

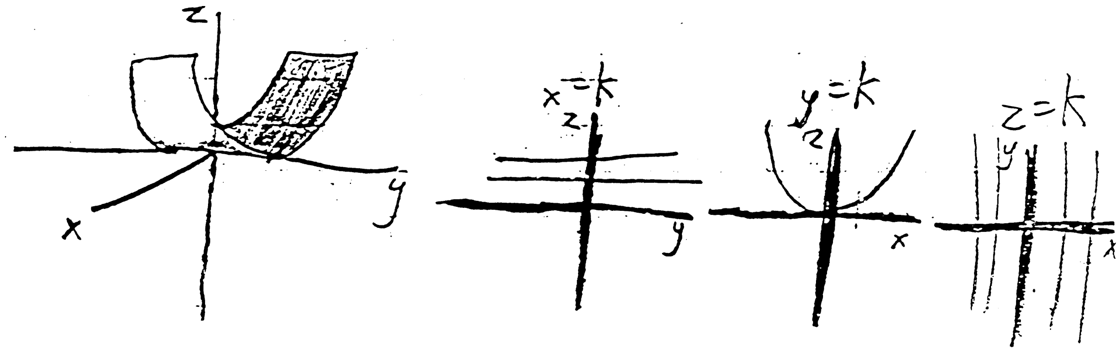
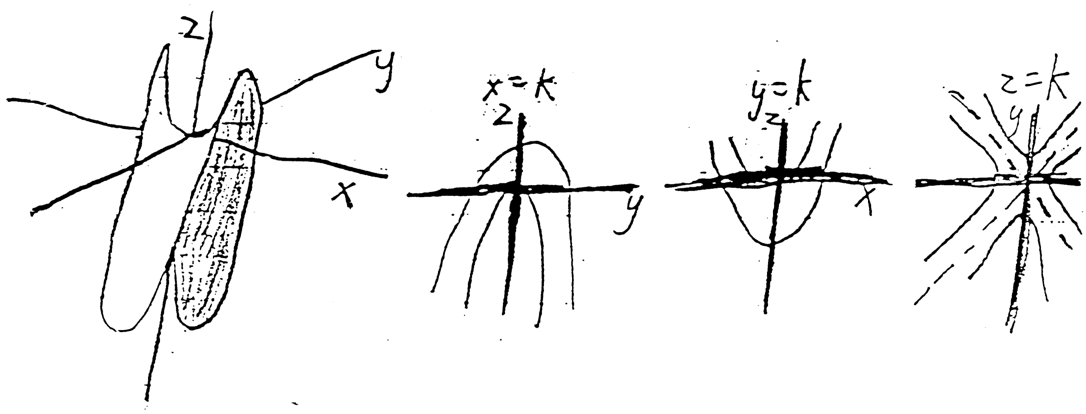


Figure 3. John's Exploration of the surface $z = x^2 + cy^2$, for $c < 0$



On the other hand, Jim's investigation of the same problem included no diagrams:

Jim: "At c>0, it is an elliptic paraboloid. In the xz and yz planes there are parabolas, and in the xy plane there is an ellipse. At c<0 it is a hyperbolic paraboloid. In the xz and yz planes there are

parabolas and in the xy plane it is a hyperbola. At c = 0, the graph
is just a parabola projected across the xz plane."

It appears that Jim's observations were entirely based on computer generated graphics. Since both 3D-Analyzer and Surfaces allow the rotation of a 3D-surface by a simple use of the mouse, the user is able to see the projection of the surface on any 2D-plane. Jim seems to have used this feature of the software programs to describe the surface. On the other hand, John's solution was visual (in Presmeg's sense). His investigation was based on his visualization of the different traces. He even spoke of the trace z = k, the equivalent of contour lines, a concept that had not been discussed yet in class. John might have used the computer, but in my judgment, he used it simply to assert his claims.

Example 2: Exploring Parametric Equations

The objective of this visualization assignment was to illustrate geometrically the position, velocity, and acceleration of a particle whose motion is described parametrically. One problem read as follows:

"Graph the following parabola:

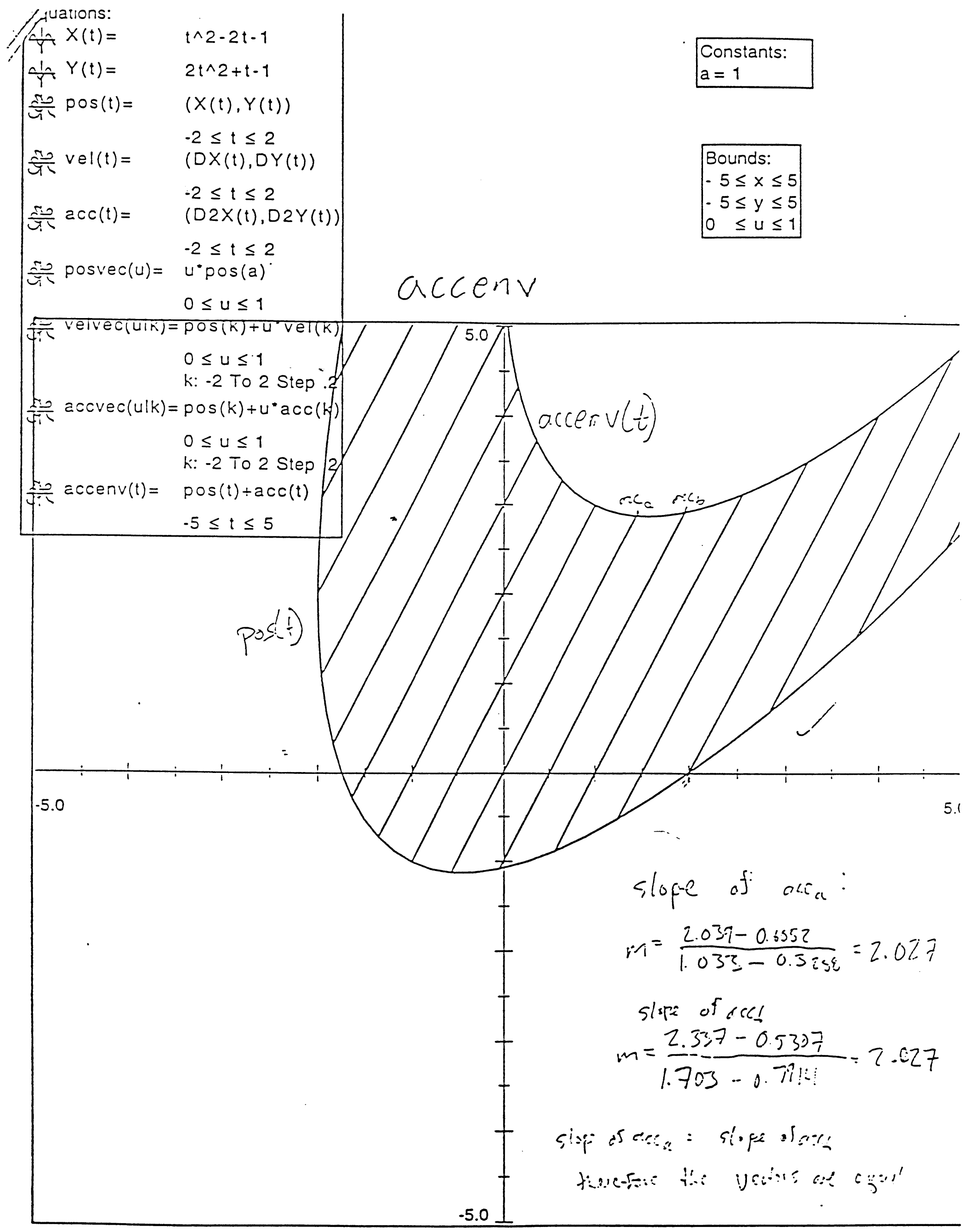
$$x(t) = t^2 - 2t - 1$$

$$y(t) = 2t^2 + t - 1$$

> and graph a family of velocity and acceleration vectors. Show that the acceleration vectors are parallel, find the vertex of the parabola (the vertex is characterized by being the point where the velocity vector is orthogonal to the acceleration vector), and find the symmetry axis."

One paper by Jay and Sam stood out since they answered the last two questions geometrically rather than analytically. Using 3D-Analyzer, the students drew the parabola and the family of velocity and acceleration vectors on one coordinate system. Since either software is capable of identifying the coordinates of any point on the screen, Jay and Sam traced the coordinates of four points belonging to two acceleration vectors, thus enabling them to calculate the slopes of these vectors (see Figure 4).

Figure 4. Jay and Sam's proof that the acceleration vectors are parallel

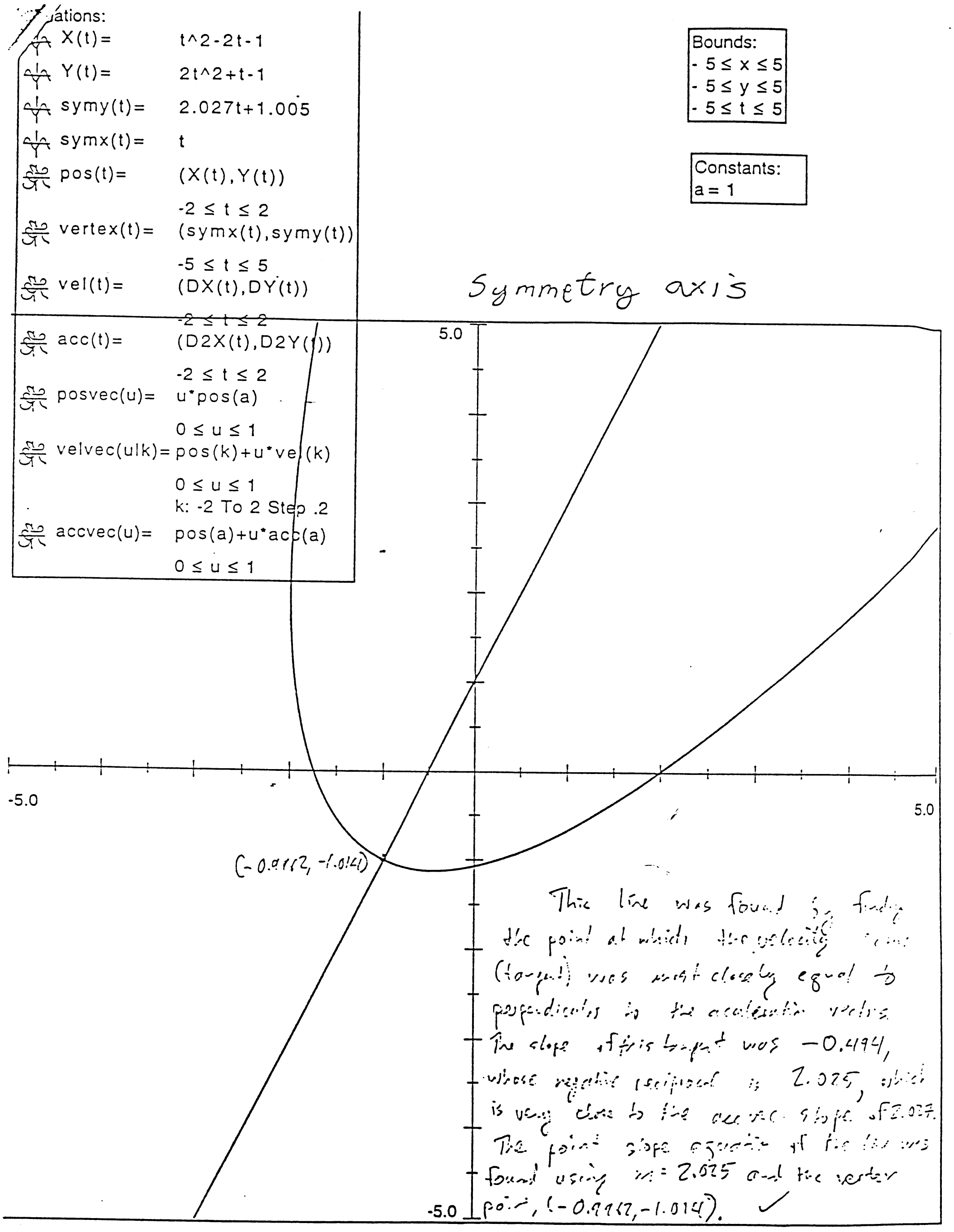


The same property was used to find the vertex. As for the symmetry axis, they wrote:

> Jay & Sam: "The line was found by finding the point at which the velocity vector (tangent) was most closely equal to a perpendicular

to the acceleration vector. The slope of this tangent was -0.494, whose negative reciprocal is 2.025, which is very close to the acceleration vector slope of 2.027. The point slope equation of the line was found using m = 2.025 and the vertex point (-. 9962, -1.014)." (See Figure 5).

Figure 5. Jay and Sam’s approach for finding the symmetry axis



Although the students did not show that <u>all</u> velocity vectors are parallel, and although an approximate value of the slope of the symmetry axis was found, their approach was original for it was purely visual. Clearly, technology was a key reason for adopting this visual strategy for solving the problem.

John and Kim also used a similar strategy in solving another problem of the same assignment. It read as follows:

"Graph the following hyperbola:

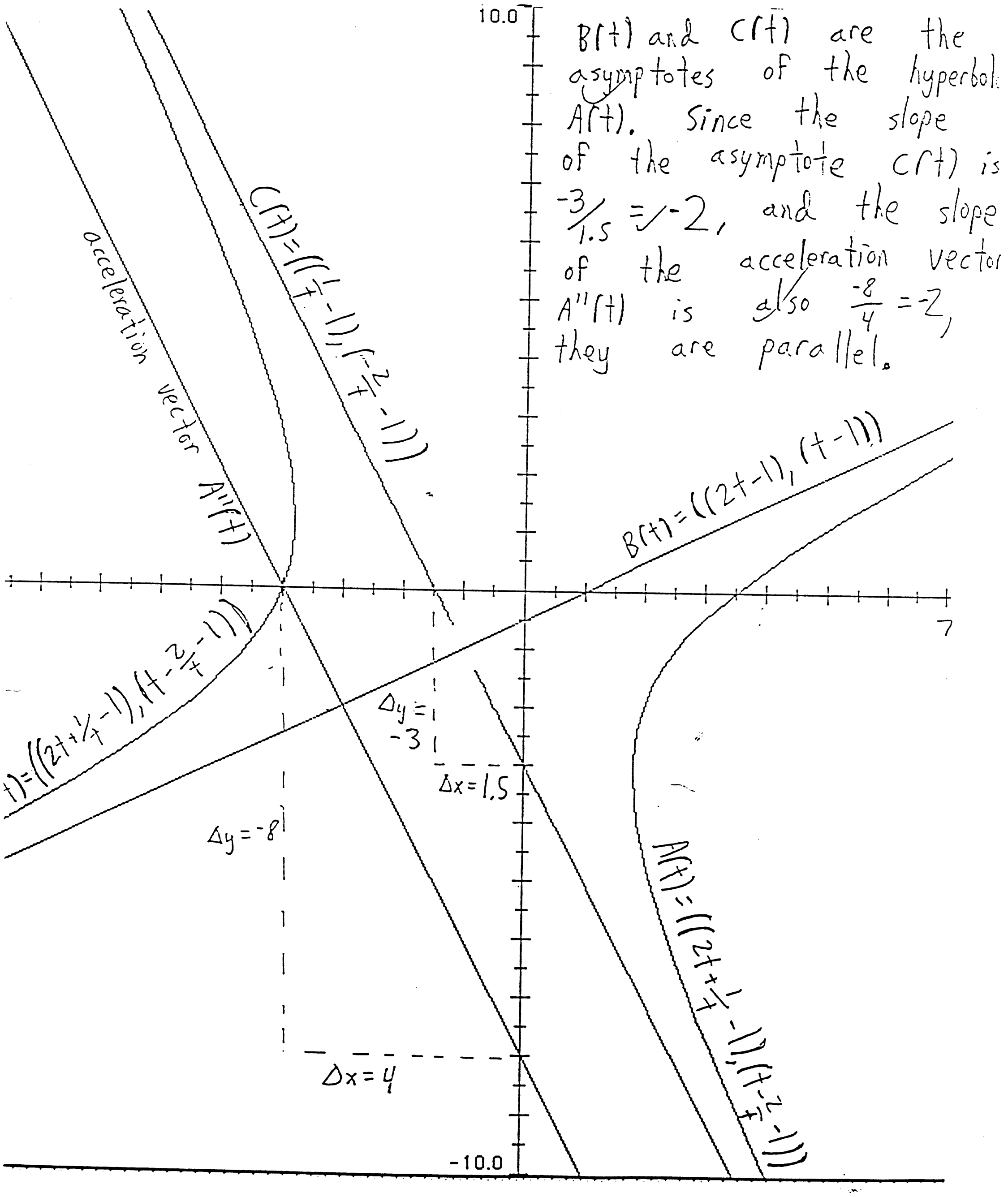
$$x(t) = 2t + \frac{1}{t} - 1$$

$$y(t) = t - \frac{2}{t} - 1$$

and show that the acceleration vector is always parallel to one of the asymptotes."

Figure 6 illustrates the similarity between the work of John and Kim and that of Jay and Sam.

Figure 6. John and Kim's proof that the acceleration vector is parallel to an asymptote

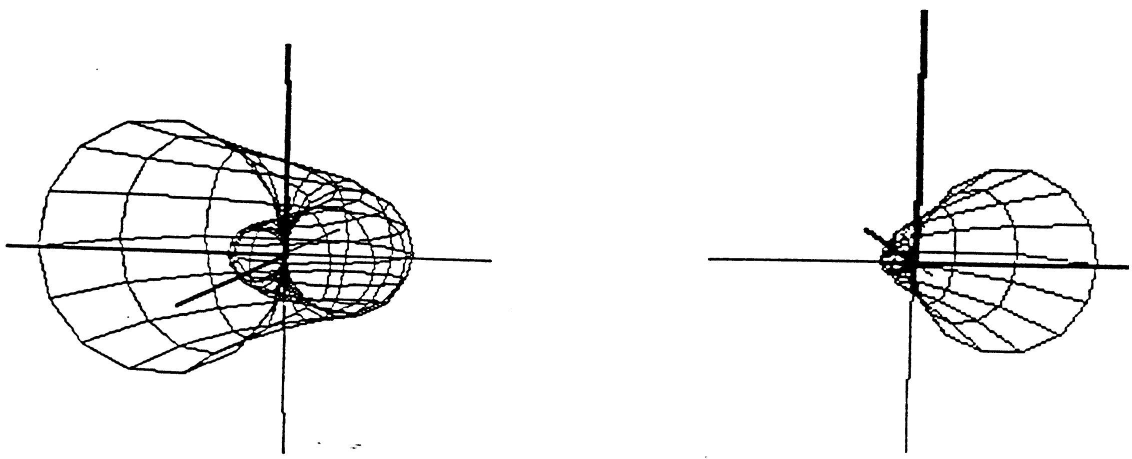


<u>Example 3: Surfaces in Spherical Coordinates</u>

In this assignment, students were asked to graph the <u>Nautilus Shell</u> whose equation in spherical coordinates is given by $\rho = (1.3)^{\theta}\sin(\varphi)$ and to experiment and comment on the role of the parameter 1.3. John presented us with the most original answer. He replaced 1.3 by c and wrote:

> John: "If c=1, the graph resembles a torus with a hole of radius zero. For c>1, the nautilus shell shape gets bigger as c increases. For 0<c<1, the shell curves inward like a clam shell. It's interesting to note how easily one can switch from a nautilus shell to a clam shell; it shows how one could have evolved into the other just by switching one "parameter" in its genes." (See Figure 7).

Figure 7. John's exploration of the surface $\rho = (1.3)^{\theta} \sin(\varphi)$



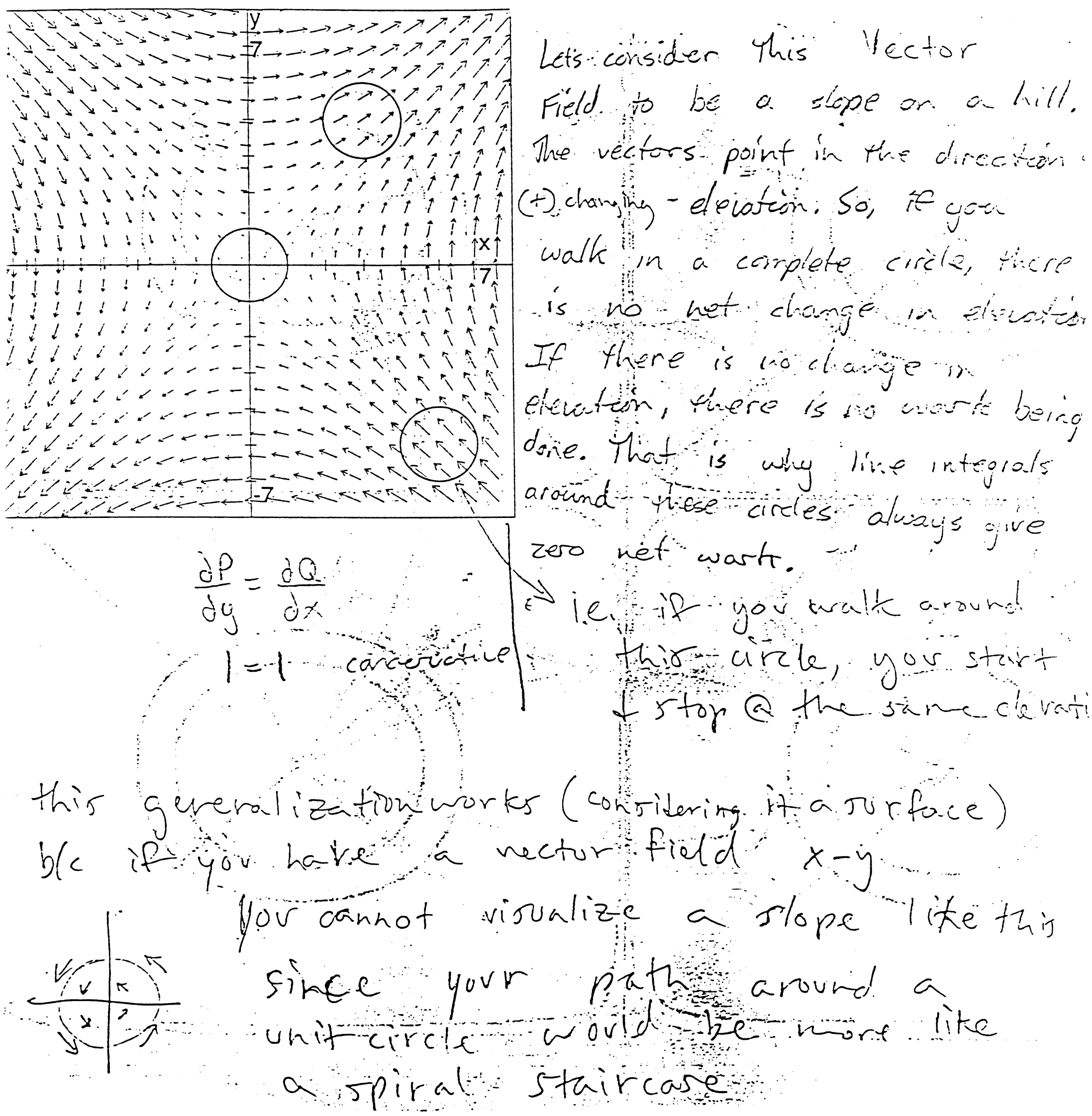
No doubt that John's stretch of imagination could not have been possible without the aid of the computer.

Example 4. Vector Fields

The objective of this assignment was to develop a visual picture of vector fields. One question asked to evaluate line integrals over paths within these fields. The instructor suggested two paths: a unit circle centered at the origin, and another centered at an arbitrary point.

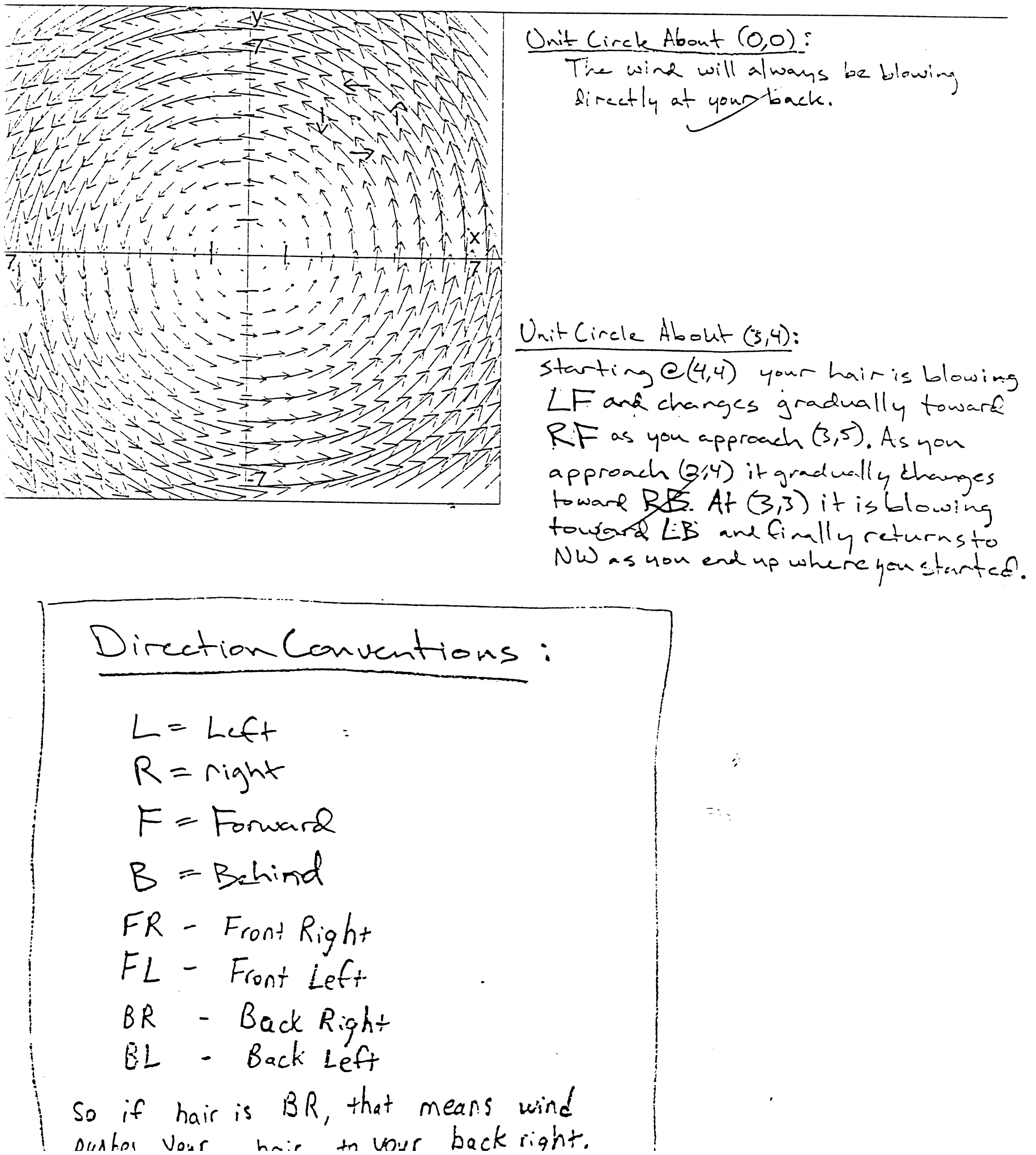
Doug and Eric chose the vector field $F = \langle x, y \rangle$. They visualized its vectors as the slopes of a hill, and argued that "line integrals around these circles always give zero net work, the reason being that if you walk in a complete circle, there is no net change in elevation." (See Figure 8).

Figure 8. Doug and Eric's exploration of the vector field *F* = (*x*,*y*)



Another question asked for a description of the effects a person would feel if he or she were to walk counterclockwise around these circles. Paul and Marc chose the vector field $F = \langle -y, x \rangle$. To make their point, they described the effect of blowing air on their hair as they move along these circles. Based upon self-defined "direction conventions" their description was original and is provided in Figure 9.

Figure 9. Paul and Marc's exploration of the vector field $F = (-y, x)$



The Students' Perspective

How helpful was the computer in the understanding of the material presented in class? How much effect did it have on the students' visualization skills? Would students consider visualization as an approach for solving a mathematical problem? Those are few questions that I will attempt to answer by quoting several comments that were compiled throughout the semester, either as parts of questionnaires, or in the form of short essays requested by the instructor.

In several homework assignments, students were asked to assess the degree to which computers were an aid in understanding and doing the homework. All students agreed that the computer exercises helped in the visualization of 3D-surfaces, "one of the most difficult aspects of the course", as many students wrote. But in other assignments as well, the role of the computer was positively evaluated. For instance, in the assignment on singularities of surfaces, technology proved to be useful because "it showed why there is no limit in certain graphs when x and y approach certain values". Some students realized that an analytic proof is needed to prove the existence or nonexistence of the limit, but the visual aspect was still important. An anonymous student wrote:

> "This graphing program was helpful in the way that we were able to determine if a function has a limit or not by looking at the graph...But to actually prove the existence or non-existence of the limit, we had to do the calculation. Yet knowing the result beforehand was helpful, because it is hard to decide which way to go just by looking at the equation."

The role of the computer was highly praised also in the assignments on linear approximations of 3D-surfaces. Many students appreciated the ability to see the tangent plane against the actual surface:

> " Looking at how the tangent planes related to their graphs was also an aid to doing and understanding the homework exercise. We could see what was happening, instead of just doing calculations." (Anonymous)

Regarding their visualization skills, a questionnaire administered after the first exam attempted to answer this and other related questions. When asked if their visualization skills have improved due to the lab work, most responded positively, but many believed that without the computer, they have a hard time visualizing a surface:

> Grace: Yes. But I can still visualize better if I actually do the problem on the computer.
>
> Jill: Yes in that when I see the equation and the graph, it makes a lot of sense. But no in that it hasn't helped me get a better idea of what a graph looks like just by seeing the equation.
>
> Tamar: I believe that I am now better able to visualize 3D-objects. However, I don't feel confident that if I am given a complex equation that I would necessarily be able to visualize it.
>
> Sam: Only in that I have realized how complicated some surfaces are. I have not improved in visualizing them myself.

When asked if visualization is necessary in the teaching and learning of multivariable calculus, all but one (Jill) thought that it is:

> Jim: I do believe that it is. If you can't visualize the problem that you are working with, then when you get an answer you don't know what it means.
>
> Al: Yes. There are a lot of things that are not very intuitive. It is a lot easier to learn things when we can actually see what we are talking about and can manipulate equations to see what factors control which features of the graph.

Notice that Jim's remarks reveal his appreciation of visualization. Yet, my observations of his work throughout the semester show that his visual capabilities are minimal. Thus, despite his weakness in these skills, Jim considers visualization an important component in the learning of multivariable calculus.

To determine if students would consider a visual approach when solving a mathematical problem, they were asked the following question: "If given the choice between doing a problem analytically or visually, what would you choose? Why? or why not?". Of the 20 students that answered, 4 preferred the visual approach to the analytical. According to these students, the visual approach gives a general idea about the problem, and later helps in the analytic solution. On the other hand, 14 students said that they prefer the analytical approach mainly because it gives exact answers. As for the remaining two, one chose both without any particular order because "they give different insights as to what is going on in the problem", and the other said, "it would really depend on the problem".

Concluding Remarks

Although the instructor of the section observed emphasized visualization in the teaching and learning process, the number of students who still preferred the analytical over the visual approach outnumber those who prefer the visual one. This is not at all surprising considering that many students might be coming from traditional schools of mathematical instruction. Consequently, their view of mathematics is entirely algebraic. On one questionnaire, Marc (a student) wrote:

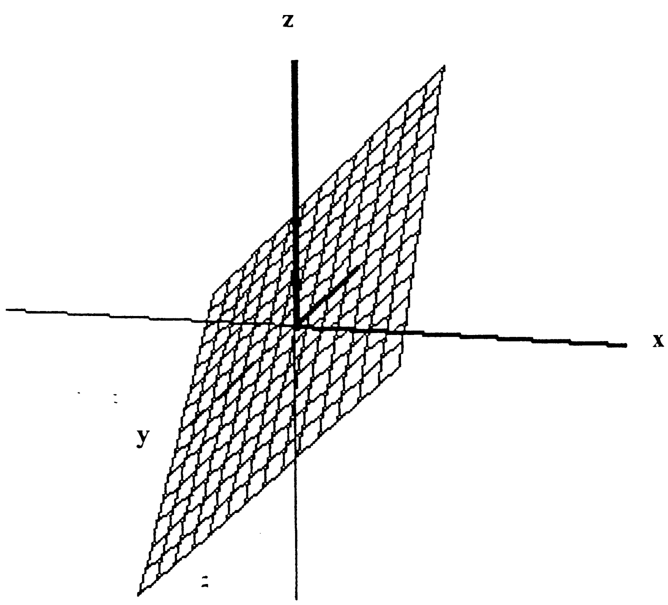
> Marc: "I think the lab does help to understand graphs more easily, but I don't think that lab should come first. I think it would be better if we learned multivariable calculus the old way before computers."

In 1991, Dreyfus has shown that mathematics instructors have recognized the power of visual thinking. Yet, in practice, implementation is lagging. Later, in 1995, Goldenberg observed that attention in mathematics classes is still devoted to symbolic expressions, with very little emphasis on the graphical or numerical representation. Thus, for the subjects of this study, it is unlikely that in a short period of time, their attitude changes in favor of multiple representation of mathematical ideas.

The difficulty with multiple representation was also clear during the interviews that I conducted with nine students of the class in the middle of the semester (see Habre (1999)). The interviews focused on 3D-surfaces. On one question, students were asked to evaluate the integral $\iint_D (x+y)dxdxy$, where D = {(x, y); -2 < x, y < 2}, and to give an interpretation for the answer (which was chosen to be zero). The instructor had discussed in class that volumes of solids may be evaluated by double integrals. Thus one interpretation of the value of the integral is to observe that, over the given domain, half of the surface $z = x + y$ lies above the xy-plane, and the other half lies below. Consequently, the volumes of the two regions enclosed by the surface cancel each other (see Figure 10). Of the nine interviewees, only three related the outcome to the geometry of the problem. All others thought that their algebra must have gone wrong:

> Jill: "Unless I made a mistake! I think I did it right...May be I should have x and y turned over."
>
> Jim: " I think I did the algebra right! I guess I could have made a mistake."
>
> Bob: "It's weird!...Um it seems it would be [wrong]...well...I am not used to seeing...if it is 2 to 2, obviously it would be zero. I think I did it right!

Figure 10. The surface $z = x + y$



It appears, therefore, that students need more time to assimilate the idea of thinking visually. However, despite the fact that they have not acquired this skill, many have come to appreciate it. The work that the students presented, and their comments throughout the semester, show that they have come to realize the importance of visualization. Furthermore, the interactivity of the software programs used lead to many creative pieces of work. Describing his experience with these programs, Paul wrote:

> Paul: "I like being able to rotate the surfaces in different directions.
> It is the next best thing to having a physical model that you can feel
> in your hands."

It remains to be seen whether, in the future, the students' attitude will change in favor of visualization (in fact, in favor of multiple representation) when solving a problem that can be done in more than one method. Technology helps in developing visualization skills, but it is necessary to convince the students that such skills are important for a broader understanding of a given problem.

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