Intuition and rigour : the role of visualization in the calculus

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1. Introduction

Visual intuition in mathematics has served us both well and badly. It suggests theorems that lead to great leaps of insight in research, yet it also can lead up blind alleys of error that deceive. For two thousand years Euclidean geometry was held as the archetypal theory of logical deduction until it was found in the nineteenth century that implicit visual clues had insinuated themselves without logical foundation: such as the implicit idea that the diagonals of a rhombus meet *inside* the figure, when the concept of "insideness" is not formally defined in the theory. Subtleties such as these caused even more pain in the calculus. So many fondly held implicit beliefs foundered when analysis was formalized: comfortable feelings about continuous functions and the ubiquity of differentiable functions took a sharp jolt with the realization that *most* continuous functions are not differentiable *anywhere*. Once the real numbers had been axiomatized through the introduction of the completeness axiom, all intuition seemed to go out of the window. It is necessary to be so careful with the statement of theorems in formal analysis that any slight lack of precision is almost bound to lead to falsehood. In such an atmosphere of fear and suspicion, visual mathematics has been relegated to a minor role – only that which can be proved by formal means being treated as *real* mathematics.

Yet to deny visualization is to deny the roots of many of our most profound mathematical ideas. In the early stages of development of the theory of functions, limits, continuity &c, visualization proved to be a fundamental source of ideas. To deny these ideas to students is to cut them off from the historical roots of the subject.

In this article is summarized research into visualization in the calculus over the last one and a half decades. It considers the strengths and weaknesses of visual imagery and relates this to the notions of intuition and rigour. It shows that visual ideas often considered intuitive by an experienced mathematician are not necessarily intuitive to an inexperienced student, yet apparently more complicated ideas can lead to powerful intuitions for the rigours of later mathematical proof. The theory of calculus is reconceptualized using the notion of "local straightness" – that a differentiable function is precisely one which "looks straight" when a tiny part of the graph is magnified. This gives a

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visual conception of the notion of a differential, which gives intuitive meaning to solutions of differential equations.

Research into mathematics education shows that students generally have very weak visualization skills in the calculus, which in turn leads to lack of meaning in the formalities of mathematical analysis. This paper, based on a number of earlier papers (Tall 1982-1989), suggests a way to use visual ideas to improve the situation.

2.1 The value of visualization

In mathematical research proof is but the last stage of the process. Before there can be proof, there must be an idea of what theorems are *worth* proving, or what theorems might be true. This exploratory stage of mathematical thinking benefits from building up an overall picture of relationships and such a picture can benefit from a visualization. It is no accident that when we think we understand something we say “oh, I *see*!”.

A good example is the famous Cauchy theorem in complex analysis that states that the integral of an analytic function round a closed curve which encloses no singularities is zero. When Cauchy stated an early version of this theorem, he thought of a complex number $z=x+iy$ analytically in terms of its real and imaginary parts. By analogy with the real case, he defined the contour integral:

$$\int_{z_1}^{z_2} f(z)\, dz$$

between two points z_1 and z_2 along a curve whose real and imaginary parts are both either monotonic increasing or decreasing. As a formal generalization of the real case, this restriction on the type of curve is natural. But if we open our eyes and looks at a picture, we see that such graphs (for x and y increasing) are a restricted set of curves lying in a rectangle with opposite corners at z_1 and z_2 (figure 1). Cauchy had to visualize the situation for a more general curve in the complex plane to give his theorem in the form for a closed curve that we know today.

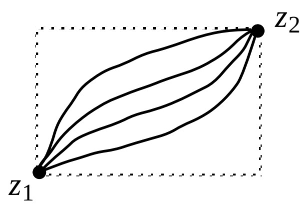


figure 1: curves with increasing x and y components

If great mathematicians need to think visually, why do we keep such thinking processes from students?

2.2 The weakness of visualization

Visualization has its distinctive downside. The problem is that pictures can often suggest false theorems. For instance, it was long believed in the nineteenth century that continuous functions have at most a finite number of points where they may be non-differentiable. The idea that a function could be continuous everywhere and differentiable nowhere was too strange to be contemplated.

Likewise, graphical methods were often used to prove analytic theorems. For instance, it was considered satisfactory to give a visual proof of the intermediate value theorem that a continuous function on an interval [a,b] passed through all the values between f(a) and f(b). The curve was considered as a "continuous thread" so that if it is negative somewhere and positive somewhere else it must pass through zero somewhere in between (figure 2).

Yet we know that the function $f(x)=x^2-2$ defined only on the rational numbers is negative for $x=1$, and positive for $x=2$, but there is no *rational* number α for which $f(\alpha)=0$. Thus visualization skills appear to fail us. Life is hard.

But not *so* hard. What has happened is that the individual has inadequate experience of the concepts to provide appropriate intuitions. In this case a possible source of appropriate intutions might be the numerical solution of equations on a computer where precise solutions are rarely found. As most computer languages represent "real numbers" only as *rational* approximations this may provide an intuitive foundation for the need to prove the intermediate value theorem rigorously.

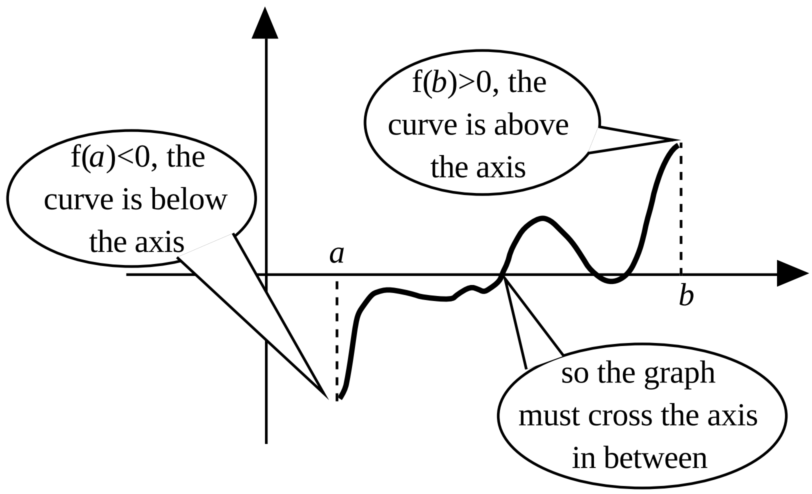


figure 2 : the intermediate value theorem

3. INTUITION AND RIGOUR

3.1 A psychological view

In his essay "Towards a disciplined intuition", Bruner characterizes two alternative approaches to solving problems:

> In virtually any field of intellectual endeavour one may distinguish two approaches usually asserted to be different. One is intuitive, the other analytic ... in general intuition is less rigorous with respect to proof, more oriented to the whole problem than to particular parts, less verbalized with respect to justification, and based upon a confidence to operate with insufficient data. (Bruner 1974, p. 99)

Some psychologists relate different modes of thinking to the two hemispheres of the brain. Glennon (1980) summarizes the findings "from many research studies" as follows:

Left hemisphere	Right hemisphere
Verbal	Visuospatial
Gestural	(including
Logical	communication)
Analytic	Analogical, intuitive
Linear	Synthetic
Sequential	Gestalt, holist
	Simultaneous &
Conceptual similarity	Multiple processing
	Structural Similarity

figure 3: charactistics of the hemispheres of the brain

Other research shows that the breakdown is not always related in this precise physical way:

> Special talents ... can reside in the right brain or in the left. Clearly what is important is not so much where things are located, but that specific brain systems handle specific tasks. (Gazzaniga 1985)

However, the principle underlying different modes of thinking remains, and we shall refer to these as the operations of the "metaphorical left and right brains", which may reside in these hemispheres in many individuals, but may be located elsewhere in others. The existence of different modes of thought suggests a distinction between intuitive thought processes and the logical thought demanded by formal mathematics. Intuition involves parallel processing quite distinct from the step by step sequential processing required in rigorous deduction. An intuition arrives whole in the mind and it may be difficult to separate the components into a logical deductive order. Indeed, it is known that visual information is processed simultaneously; only the *result* of this processing is made available to the conscious self, not the process by which the gestalt is formed (Bogen 1969, Gazzaniga 1974). Taken to extremes, this suggests that the logic of mathematics may not be well served by an intuitive approach.

On the other hand, a purely logical view is also cognitively unsuitable for students:

> We have all been brainwashed by the undeserved respect given to Greek-type sequential logic. Almost automatically curriculum builders and teachers try to devise logical methods of instruction, assuming logical planning, ordering, and presentation of content matter ... They may have trouble conceiving alternative approaches that do not go step-by-step down a linear progression ... It can be stated flatly, however, that the human brain is not organised or designed for linear, one-path thought.
> (Hart, 1983, page 52)

> ... there is no concept, no fact in education, more directly subtle than this: the brain is by nature's design, an amazingly subtle and sensitive pattern-detecting apparatus.
> (ibid. page 60)

> ... the brain was designed by evolution to deal with natural complexity, not neat "logical simplicities" ... (ibid. page 76)

There is much evidence to show that the most powerful way to use the brain is to integrate *both* ways of processing: appealing to the (metaphorical) right brain to give global linkages and unifying patterns, whilst analysing the relationships and building up logical inferences between concepts with the left. This requires a new synthesis of mathematical knowledge that gives due weight to both ways of thought. In particular, it needs an approach which appeals to the intuition and yet can be given a rigorous formulation.

3.2 Geometric concepts need not be intuitive

One of the reasons why the teaching of the calculus is in disarray is that concepts which expert mathematicians regard as intuitive are not "intuitive" to students. The reason is quite simple. Intuition is a global resonance in the brain and it depends on the cognitive structure of the individual, which in turn is also dependent on the individual's previous experience. There is no reason at all to suppose that the novice will have the same intuitions as the expert, even when considering apparently simple visual insights. Mathematical education research shows that students' ideas of many concepts is not what might be expected.

For example, because the formal idea of a limit proves difficult to comprehend in the initial stages of the calculus, it is usually introduced through visual ideas, such as the derivative being seen as the limit of a sequence of secants approaching a tangent.

Empirical research shows that the student has a number of conceptual difficulties to surmount. For instance, Orton (1977) reported the following responses from 110 calculus students:

> When questioned what happens to the secants *PQ* on a sketched curve as the point Q_n tends towards *P* on the circle, 43 students seemed incapable, even when strongly prompted, to see that the process led to the tangent to the curve:

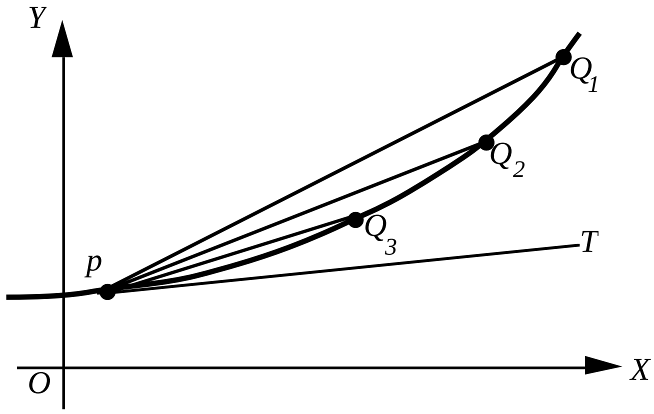


figure 4 : secants and the limiting tangent

> There appeared to be a considerable confusion in that the secant was ignored by many students, they appeared only to focus their attention on the chord *PQ*, despite the fact that the diagram and explanation were intended to try to insure that this did not happen... Typical unsatisfactory responses included : "the line gets shorter", "it becomes a point", "the area gets smaller"...

A similar question was asked in Tall (1986a):

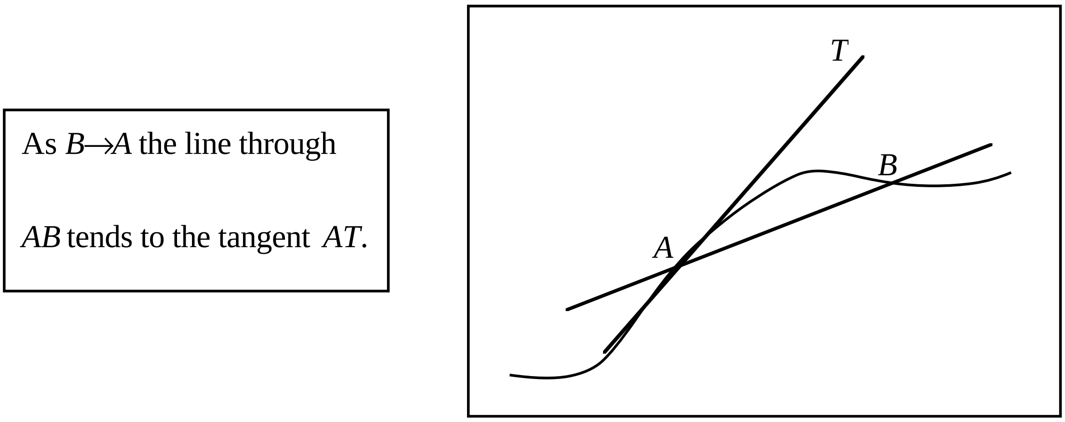
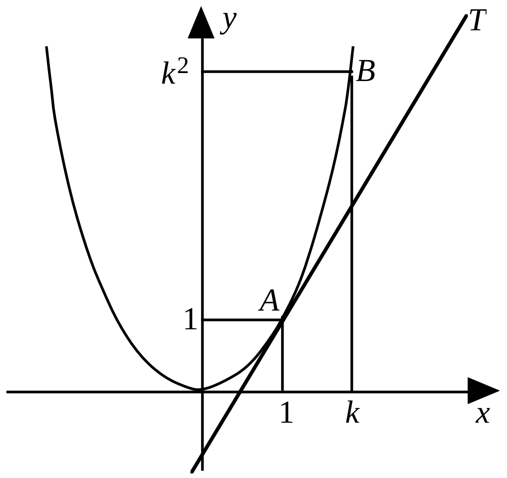


figure 5 : a secant "tending" to a tangent

Of a sample of nine 16 year old students interviewed in depth (as part of a larger project), four said the statement was "true" but linked the symbol $B \to A$ to *vector notation* and visualized *B* as moving to *A*, *along the line BA*. For them the line (segment) *BA* certainly "tends" to the tangent, but in a completely unexpected sense. Meanwhile, another student considered the statement "false" because "way off at infinity the line *AB* and tangent *AT* would always be a long way apart no matter how close *A* and *B* become". Thus it is possible also to have an "incorrect" response for a very sensible reason.

In Tall (1986), a question to investigate the intuitive nature of the limiting process was given to 160 students about to start a calculus course in the UK, of whom 96 had already had some calculus experience (figure 6).

Only 16 students (10%) obtained both the value $k+1$ for the gradient of AB and 2 for the tangent, whilst 44 (24%) obtained $(k^2-1)/(k-1)$ and 2. After the first two months of the calculus course, the numbers changed hardly at all to 17 (11%) and 38 (24%) respectively. Of these, only *one* student on the pre-test (who had already had calculus experience) and *one* student on the post-test allowed k to tend to one to find $k+1$ tends to 2. On interviewing other students, it was clear that no limiting idea occurred to them. One who found the gradient of AB to equal $k+1$ could see visually that the gradient of AT was about 2. It was suggested that as B got close to A, so k would get close to 1 and $k+1$ would get close to 2. I well remember the amazement on his face when he realized this for the first time.



On the graph $y=x^2$, the point A is (1,1), the point B is (k,k^2) and T is a point on the tangent to the graph at A.

(i) Write down the gradient of the straight line through A, B ...

(ii) Write down the gradient of AT ...

Explain how you might find the gradient of AT from first principles.

figure 6 : is the limit concept intuitive?

In this experiment a *spontaneous* limit concept did not occur to *any* pupil with no calculus experience. This gives no support to the idea that the geometric limit is an intuitive concept.

On the contrary, many other research investigations point to serious conceptual problems with the limit concept (Schwarzenberger & Tall 1977, Cornu 1981, Tall & Vinner 1981, Sierpiń ska 1987). Students have difficulties because of the language, which suggests to them that a limit is “approached” but *cannot be reached*. They have difficulties with the unfinished nature of the concept, which gets close, but never seems to arrive. They have even more difficulties handling the quantifiers if the concept is defined formally.

3.4 A new kind of intuition for the calculus

If we are to solve the chasm that occurs in student's understanding of the calculus, then I hypothesise that we must find a way that is cognitively appealing to the student at the time the study commences, yet has within it the seeds for understanding the formal subtleties that occur later. My analysis of the difficulty is that we will certainly not do this by making the concepts simpler. *The alternative is to make them more complicated!*

This is not as foolish as it sounds. The idea is to appeal to the visual patterning power of the metaphorical right brain, in such a way that it lays down appropriate intutions to service the logical deductivity of the left.

The reason why nineteenth century mathematicians found the concept of an everywhere continuous, nowhere differentiable function unintuitive was simply that they had not met a friendly example. Nor, I believe, have many of our current generation of professional mathematicians. On one occasion I asked all the members of an internationally known mathematics department if they could furnish me with a simple proof of the existence of an everywhere continuous nowhere differentiable function. None of them could do this on the spot, though two could name a book where a proof could be found and one was even able to give the page number! I was equally unable to formulate such a proof at the time. If we professionals are so unable to give a meaningful explanation of a concept, what hope is there for our students?

The answer lies in effective use of visualization, *to give intuition for formal proof*, as I shall now show.

4. A LOCALLY STRAIGHT APPROACH TO THE CALCULUS

Given that the concept of limit seems such an unsatisfactory cognitive starting point for the study of calculus, and attempts at making it geometrically "intuitive" also fail, we need a subtly different approach. This is possible through an amazingly simple visual device. We know the gradient of a straight line $y=mx+b$ is just the change in y-coordinate divided by the corresponding change in x-coordinate, but this fails for a curved graph. The answer is to *magnify the picture*. If a sufficiently tiny part of the graph is drawn highly magnified, then *most of the familiar graphs look (locally) sraight.*

4.1 Local straightness

Drawing graphs accurately by hand is a major activity. But once students have some experience of drawing graphs, a graph-plotting program can be used to magnify the picture. This is best done with a plotter with (at least) two graph windows: one for the original scale graph, the other to see a magnified smaller portion (figure 7).

Given a little time to experiment students will hypothesise that the more a graph is magnified, the *less curved* it gets. When it is suitably highly magnified, it will look *locally straight.*

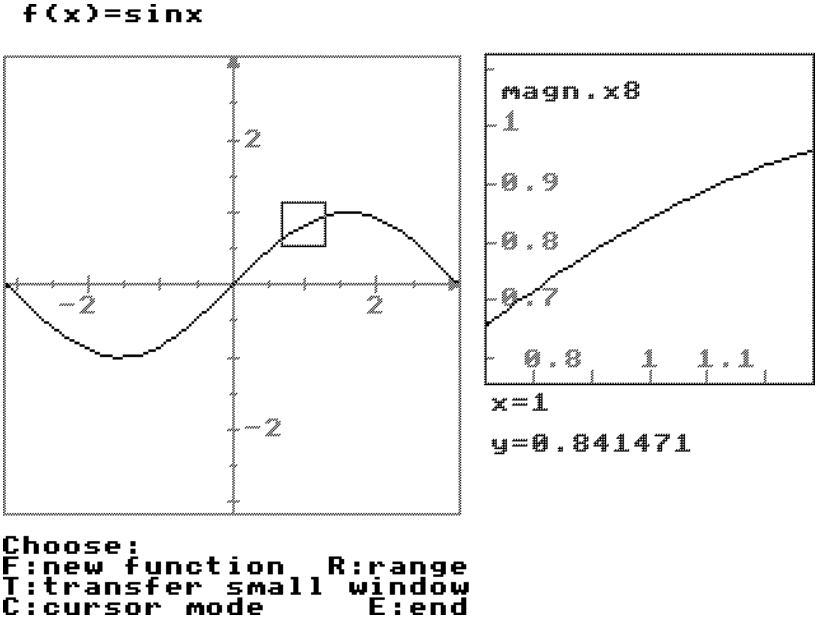


figure 7: a locally straight curve (magnified a little)

These students now have *a significantly different mind set* from traditional students. They are able to cast their eye along a graph and see its changing gradient. Their visual intuition is sharper.

4.2 Non-locally straight graphs

Students just given simple examples of locally straight graphs are likely to be dangerously misguided. For, just as nineteenth century mathematicians were convinced by their limited experience that “most” graphs are differentiable “almost everywhere”, limited unguided experimentation can easily lead to the belief that all graphs are locally straight.

So we must make the experience more complicated – *immediately* – before the mind is set. If asked to suggest graphs which are *not* locally straight, my experience is that students find it very difficult to make the first step. But once one or two examples are given, the floodgates open. It is now my preference, *in the first lesson*, to look at graphs like $y=|x|$ or $y=|\sin x|$, or $y=|x^2-x|$ to see that they have “corners” which magnify to two half-lines with different gradients meeting at a point. It is also easy to jazz these up a little to add a tiny graph like $y=|\sin 100x|/100$ to a smooth graph to get, say

$$y = \sin x + |\sin 100x|/100$$

which looks smooth, like $\sin x$, to a normal scale, yet has corners when magnified by a factor 100 or so.

It is also interesting to focus on points where a graph may oscillate strangely, such as f(x)=x sin(1/x) at the origin (with f(0)=0), or even g(x)=(x+|x|)sin(1/x). The latter is locally straight in one direction at the origin, but oscillates wildly in the other.

4.3 Nowhere differentiable functions

Even exhibiting curves with corners gives inadequate intuition. We must take our courage in both hands and go the whole way. The *Graphic Calculus* software (Tall 1986, Tall *et al* 1990) includes a model of an everywhere continuous, nowhere differentiable function called *the blancmange function* after a custard pudding with a similar shape. The function is simply so wrinkled that, wherever it is magnified it still looks wrinkled (figure 8).

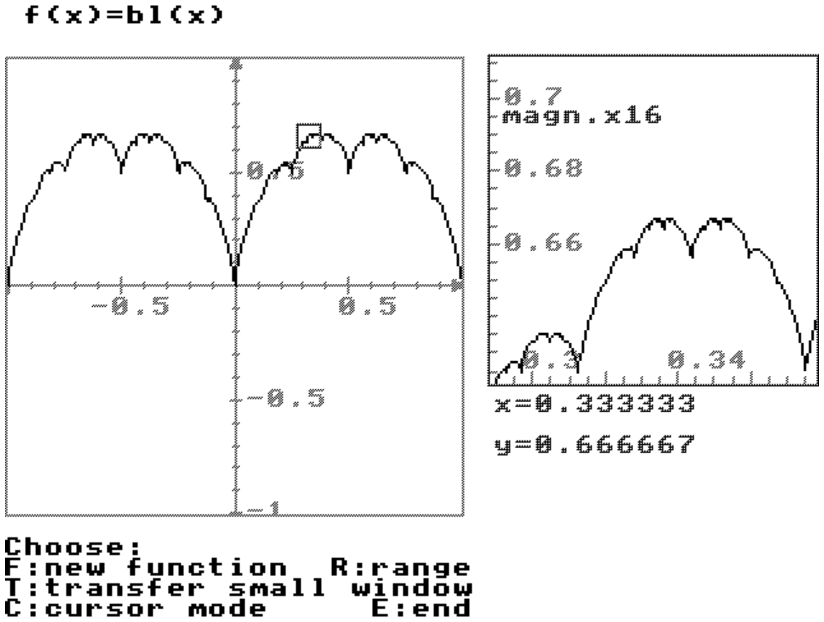


figure 8: a highly wrinkled function that is nowhere locally straight

The power of this function, and the recursive way that it is defined, is that it is easy to give an intuitive proof as to why it is nowhere differentiable. The argument (given in Tall 1982) is as in figure 9.

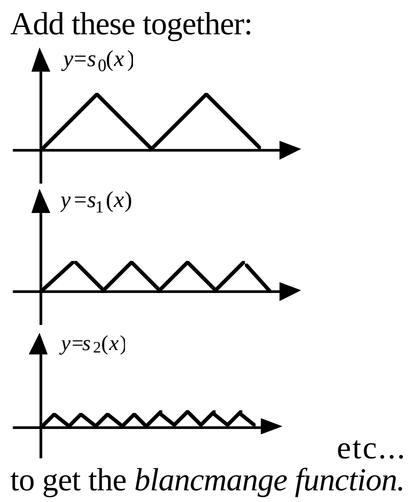


figure 9

Similarly, the sum can be broken down into the sum of the first two saw-teeth, plus the sum of the third, fourth, fifth etc. The sum of the first two is the second approximation to the blancmange, the sum of the remainder is a quarter-size blancmange. In general the blancmange can be seen as being the *n*th approximation with a $1/2^n$ size blancmange added. This is why it is so wrinkled. It has blancmanges growing everywhere!

Once it is understood that an everywhere wrinkled graph $y=\text{bl}(x)$ can be constructed, it is easy to see that the graph

$$\text{n}(x)= \text{bl}(1000x)/1000$$

is a very tiny wrinkle indeed (it is smaller than 1/1000, and only shows up under magnification of a 100 or so). In fact $y=\sin x$ looks just like $y=\sin x+\text{n}(x)$ but the first is locally straight and the second looks wrinkled under high magnification.

Thus, by visualizing, we have broken the fetters of visualization. We can envisage two graphs, which look the exactly the same at normal magnification, one differentiable *everywhere*, one differentiable *nowhere*...

4.4 Enactive gradients

Once the idea of local straightness is established, all the standard derivatives can be conjectured by *looking* at the gradient of the graphs. In addition to using the computer, it is possible to carry the action out using a simple tool - designed by the School Mathematics Project in England – and called the *gradient measurer*. This is just a circular piece of transparent plastic with a diameter marked, which can turn around its centre. It is affixed to another transparent piece of plastic (by slotting into lips which overlap the diameter) on which is marked a vertical ruler in units, one unit horizontally away from the centre.

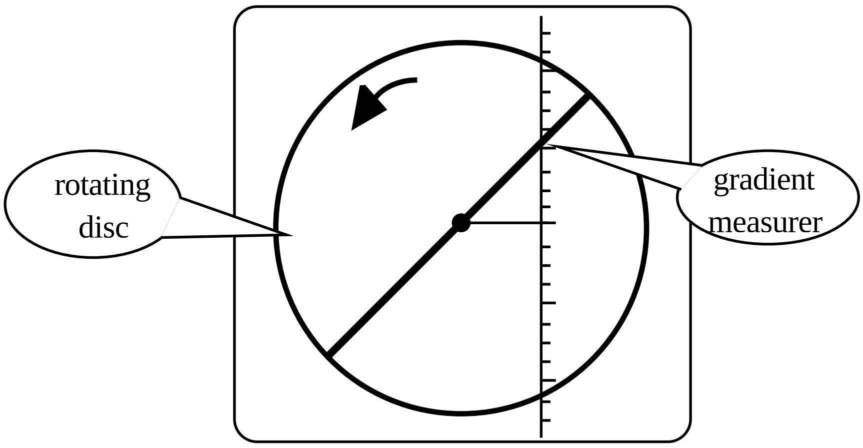


figure 10 : a tool for measuring gradients

Using the gradient measurer, a student can place it over a point on the graph, rotate the disc until the marked diameter is visually in the direction of the graph at that point, then read off the gradient. Thus she or he can move the measurer along the graph and enact the changing gradient, as well as obtain approximate numerical results. (Presupposing, of course, that the graph is a *faithful* representation of the gradient, both in terms of having the same *x*- and *y*-scale and also not having any tiny wrinkles that cannot be seen at this scale.)

4.5 Computer generated gradients

It is an easy matter to program the computer to draw the numerical gradient $\frac{f(x+c)-f(x)}{c}$ for a fixed, but small, value of *c*. This can be done in such a way that the student can superimpose a conjectured graph to compare with it. In this way it is possible to conjecture what the derivative of simple functions will look like. All the standard functions can be investigated in this way.

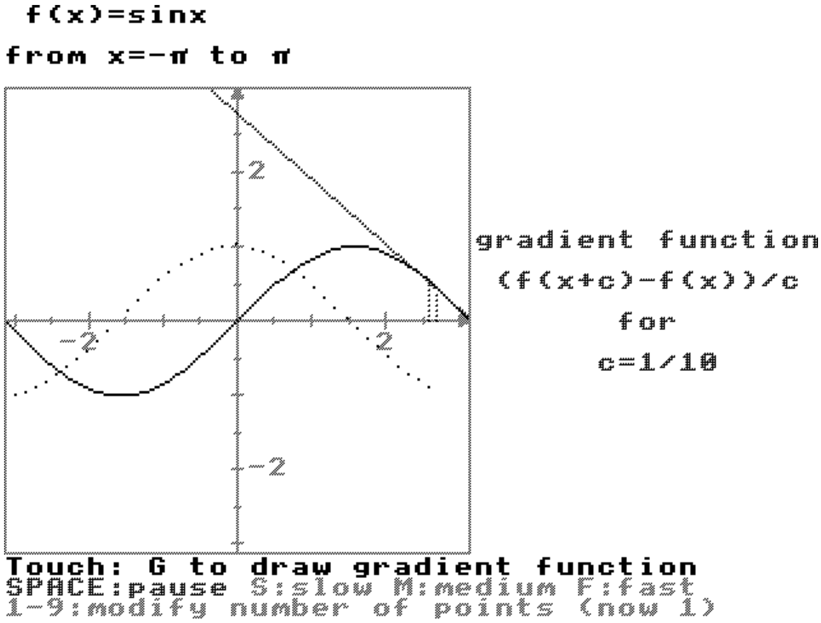


figure 11 : visualizing the global gradient function

By using this approach, the seed is sown that the gradient changes as a *function* of *x*…

5. DIFFERENTIAL EQUATIONS : UNDOING LOCAL STRAIGHTNESS

5.1 (First order) differential equations

First order differential equations tell us the gradient d*y*/d*x*=f(*x*,*y*) of a solution curve. Because the curve has a derivative (equal to f(*x*,*y*) at (*x*,*y*)), it *must* be locally straight. So an approximate solution may be constructed by building up curves out of tiny line segments having the direction specified by the differential equation.

5.2 sketching solutions in an enactive computer environment

Sketching solutions of a differential equation

$$\frac{dy}{dx} = f(x,y)$$

by hand is a complicated process. But the tasks may be shared with a computer. The *solution sketcher* (Tall 1989) does this by showing a short line segment centred at a point (x,y) with gradient f(x,y) (figure 12).

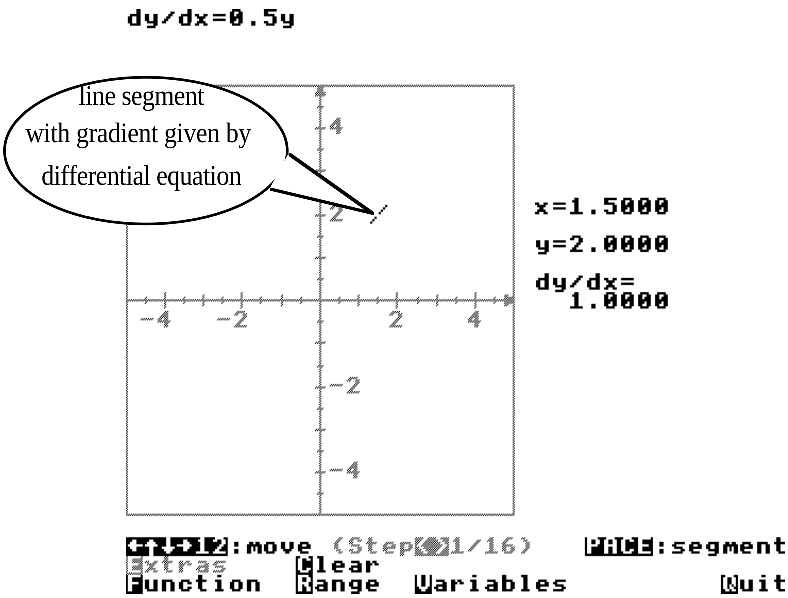


figure 12: the solution sketcher

The line segment may be moved around, either using the mouse or cursor keys. As this happens the segment takes up the direction given by the differential equation dy/dx=f(x,y). Clicking the mouse or touching the SPACE bar leaves a copy of the segment at the current position, allowing the user to build up a solution by following the direction specified in an *enactive way*. By this I mean that the user carries out the physical act of following a solution curve, whilst the computer calculates the gradient.

All the usual formal theory about the existence and uniqueness of solutions arises through enacting the solution process physically, providing powerful intuitions. There is a unique solution through every point (x,y) and it continues as long as the equation continues to specify the required direction.

Figure 20 shows a solution constructed in this way. It is superimposed on a whole array of line segments whose gradients are given by the differential equation, showing the *global* trends of other possible solutions.

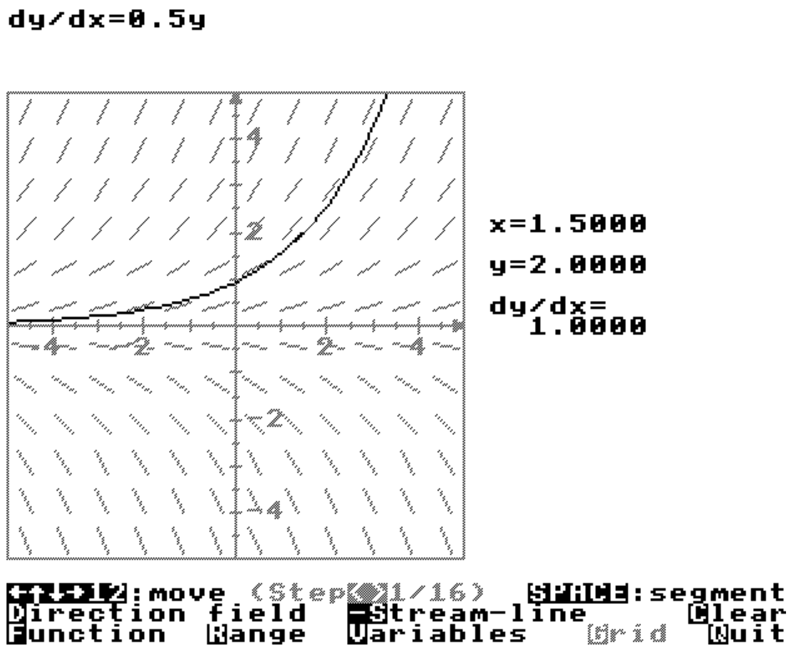


figure 13 : constructing a solution of a differential equation

5.3 Practical activities - enactive visual

Although the gradient measurer (figure 10 above) is not a viable tool to use for drawing solutions, it can be used to check that a solution has the required gradient everywhere, thus confirming that different ways of performing the same process give similar results. Intuition is always supported more strongly when different methods tell the same story.

5.4 Functions differentiable once but not twice...

Solving differential equations such as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{bl}(x)$$

where the right hand side is continuous, but not differentiable, gives a solution curve y=F(x) whose gradient satisfies F′(x)=bl(x). Thus the function F is differentiable once, but not twice. Repeating the process several times can give a function which is differentiable n times but not n+1, enabling the mental imagery to be developed to encompass such functions.

5.5 Higher order differential equations

Visualization of second order differential equations can proceed in a similar way, but it no longer gives unique solutions through each point. Again it gives intuitive support to later formalities. The original *Graphic Calculus* software (Tall 1986) uses the theory that a second order differential equation, for instance,

$$\frac{d^2y}{dx^2} = -x$$

can be rewritten using v=dy/dx to get two simultaneous first order differential equations:

dv/dx=-x, dy/dx=v.

The solutions are simply curves in (x,v,y) space where the tangent direction (dx,dv,dy) satisfies

dv=-xdx, dy=vdx.

Thus a solution is a curve in (x,v,y) space found by following this tangent direction and this can be done by computer software to produce pictures like those of the parametric curves above.

6. INTEGRATION

6.1 The "area" under a graph

Integration is the idea of "cumulative growth", which is usually seen as calculating the area "under" a curve. Thus can be performed using thin rectangular strips, or methods such as the trapezium or Simpson rule. Visually, by taking a larger number of strips, it becomes apparent that sum of strip-areas is likely to give the value of the area under the curve. However, flexible software may be used for all kinds of investigations to give more powerful intuitions. For instance, most students, and not a few teachers, believe that the area is "positive above, negative below", but if the sign of the step is negative, the reverse is true. Figure 21 has a negative step and a negative ordinate in the range working backwards from 2π to π, giving a *positive* result when the curve is below the x-axis. Although this seems more complex than just giving a simple rule, it easily provides a complete mental picture of the four possible combinations of sign of step-direction and ordinate, with the dynamic movement giving powerful intuitions linking with signed arithmetic.

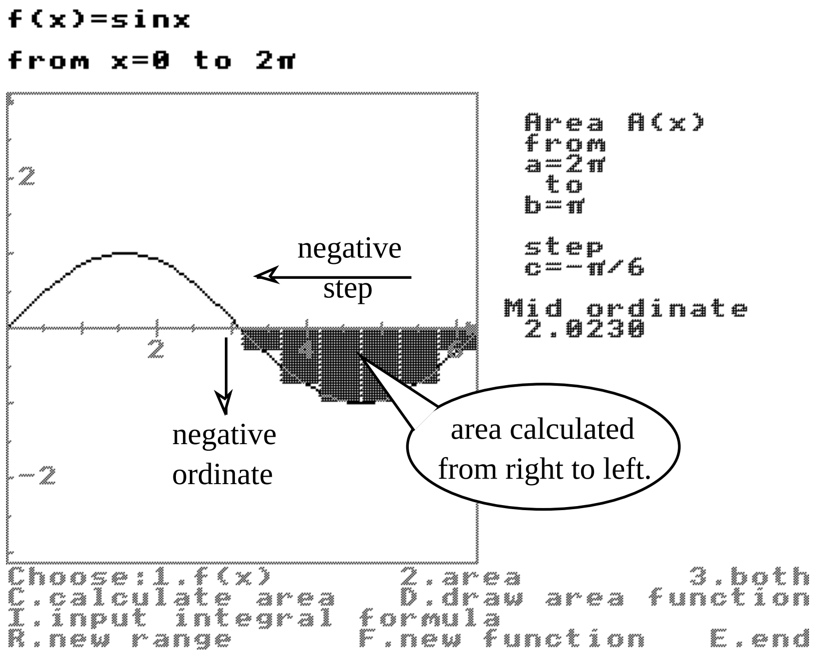


figure 14: negative step and negative ordinate gives a positive area calculation

6.2 The "area-so-far" function

The area-so-far graph may be drawn by plotting the value of the cumulative area calculations from a fixed point to a variable point. Figure 22 shows the superimposition of two area calculations, first from x=0 to x=-5 with the negative step -0.1, then from x=0 to the right with positive step 0.1. Notice the cubic shape of the dots of the area curve, which experts will recognize as $y=x^3/3$. But we rarely consider this for *negative x*.

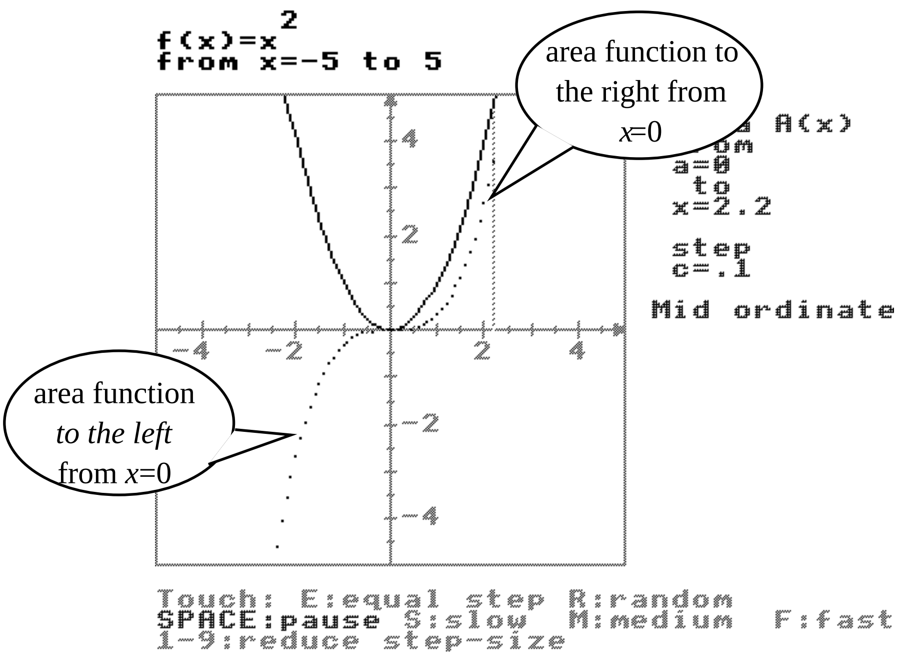


figure 15: the area function from x=0 in both directions

7. UNDOING INTEGRATION : THE FUNDAMENTAL THEOREM

7.1 Numerical Gradients and Areas as Functions

There are many programs around that will draw numerical gradients or numerical areas, but these are usually just a sequence of points plotted on the screen. An area graph plotted as in figure 15 simply records the cumulative area calculations pictorially and does not remember them in any way. But the area under a graph of a given function, from $x=a$ to $x=b$, with a given step, say h, is calculated by a straightforward computer procedure. Given a suitably fast processor this can be calculated almost instantaneously and may be considered as a function depending on the graph and the values of a,b, h.

In the Function Analyser (Tall 1989) the expression:

area(*expr*, a, b, h)

is interpreted as the area under the graph given by the expression *expr*, from $x=a$ to $x=b$ using the mid-ordinate approximation with step-width h. Thus:

area(sin$x$, 0,x,0.1)

is the area under $y=\sin x$ from 0 to x with step-width 0.1. Such is the power of the Archimedes computer in British schools that this graph can be drawn in less than three seconds with 100 intermediate points each requiring an area calculation for up to 50 strips.

The numerical area is now truly a *function*, which may be numerically differentiated like any other function. Figure 23 shows the area function area(bl(x),0,x,s) under the blancmange function from 0 to x using strip-width s=0.05. (The area function is the rather bland looking increasing function, not the pudding-like blancmange).

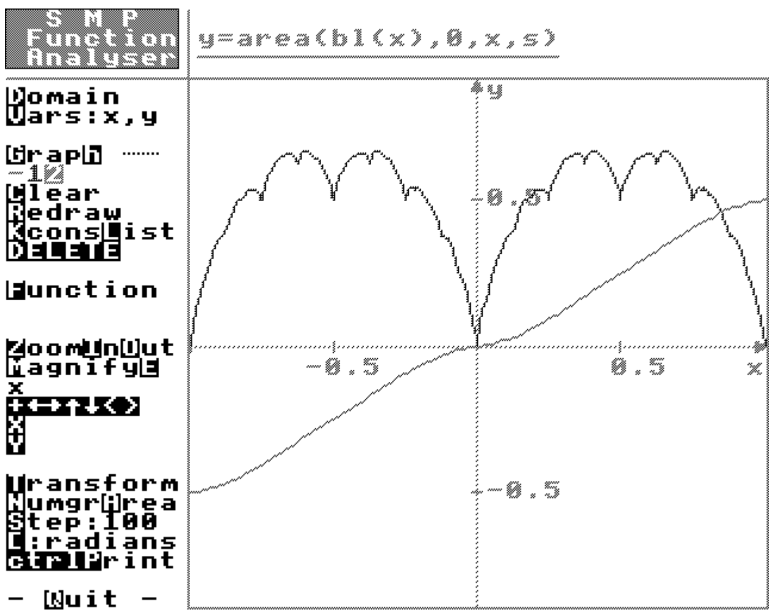


figure 16: the area function for the blancmange and its derivative

Of course, this graph is not the exact area function, but it is a good-looking approximation to it – as good as one could hope to get on a computer screen. Notice that it looks relatively smooth – which it is, if one ignores the pixellation problem – because the derivative of the exact area is the blancmange function. The exact area function for the blancmange is a function which is differentiable everywhere once and nowhere twice!...

The other graph in figure 17, by the way, may look like the blancmange function, but it is actually the graph of

area(bl(x),x,x+w,h)/w

(the numerical derivative of the numerical area function for the blancmange). Even though the blancmange function is nowhere differentiable, its area function is quite smooth and differentiable everywhere precisely once.

7.2 Stretching the imagination for the fundamental theorem of the calculus

One way to visualize the fundamental theorem is to imagine a tiny part of the graph stretched horizontally (figure 17). In many cases the graph of a function stretches out to look flat – the more it is stretched, the flatter it gets.

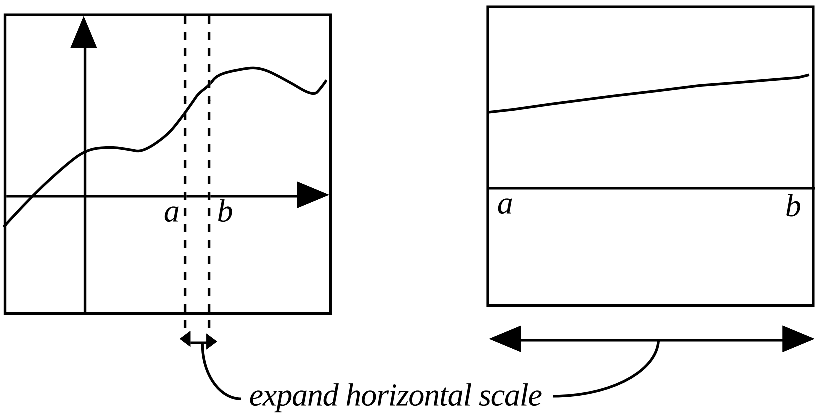


figure 17 : stretching a graph horizontally

This is easy to see with a graph drawing program using a thin x-range and a normal y-range to stretch the graph horizontally in a standard graph window.

Figure 18 shows that if the area from a fixed point a to a variable point x is A(x), then the area from a to x+h is A(x+h), so the change in area from x to x+h is A(x+h)–Ax).

If the strip from x to x+h is approximately a rectangle width h, height f(x), then its area is

$$A(x+h)-A(x) \approx f(x)h$$

giving

$$\frac{A(x+h)-A(x)}{h} \approx f(x).$$

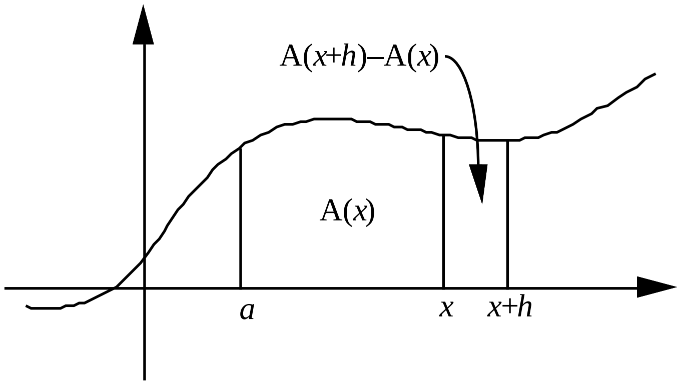


figure 18 : the change in area

As h gets smaller, the graph gets pulled flatter, the approximation gets better, giving intuitive foundation for the **fundamental theorem of the calculus**:

$$A'(x) = f(x).$$

7.3 From intuition to rigour

A formal proof of the fundamental theorem requires the notion of continuity. This notion is usually confused intuitively by students, teachers and mathematicians alike. Ask anyone with some knowledge of the concept to explain what it means and the likely answer is that is is a function whose graph has "no gaps" – its graph can be drawn "without taking the pencil off the paper" etc, etc. (Tall & Vinner 1981). These ideas are *not* the intuitive beginnings of continuity but of "connectedness" which is mathematically linked, but technically quite different.

Continuity can be seen to arise from the horizontal stretching of graphs in the fundamental theorem. Consider a simplified model of what is happening in stretching the graph to confine it within a horizontal line of pixels. Suppose that graph picture has middle x-value $x=x_0$ and the point $(x_0,f(x_0))$ on the graph is in the middle of a pixel whose upper and lower values are $f(x_0)–e$ and $f(x_0)+e$. To fit the graph in a horizontal line of pixels means finding a small x-range from x_0-d to x_0+d so that for any x in this range the value of $f(x)$ lies in the "pixel range" between $f(x_0)–e$ and $f(x_0)+e$ (figure 19).

This gives the formal definition of continuity:

> The function f is continuous at x_0 if, given *any* specified error $e>0$, there can be found a (small) distance d such that whenever x is between x_0-d and x_0+d, so $f(x)$ is between $f(x_0)–e$ and $f(x_0)+e$.

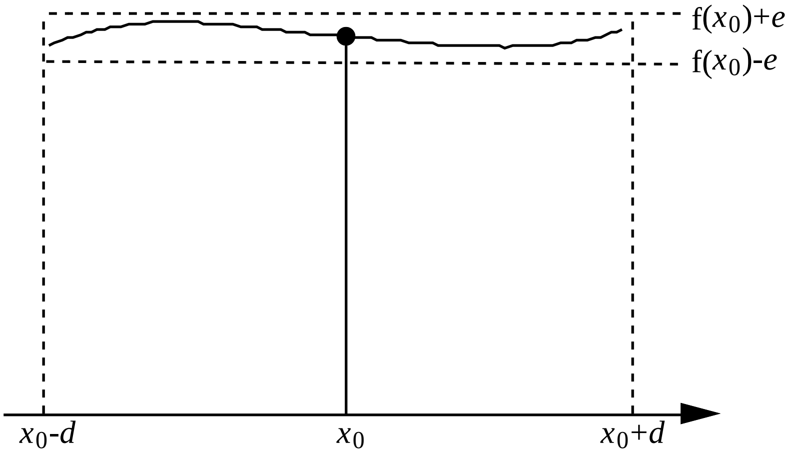


figure 19 : the concept of continuity through horizontal stretching

The ramifications of this definition take months, even years, to understand in full, but it has an appealing intuitive foundation: *a continuous function is one whose graph has the property that any suitably tiny portion stretched horizontally will pull out flat.*

8. Conclusions

By introducing *suitably complicated* visualizations of mathematical ideas it is possible to give a much broader picture of the possible ways in which concepts may be realized, thus giving much more powerful intuitions than in a traditional approach. It is possible to design enactive software to allow students to explore mathematical ideas with the dual role of being both immediately appealing to students and also providing foundational concepts on which the ideas can be built. By exploring examples which work and examples which fail, it is possible for the students to gain the visual intuitions necessary to provide powerful formal insights. Thus intuition and rigour need not be at odds with each other. By providing a suitably powerful context, intuition naturally leads into the rigour of mathematical proof.

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